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Multifractal analysis of the Brjuno function

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Abstract The Brjuno function B is a 1-periodic, nowhere locally bounded function, introduced by Yoccoz because it encapsulates a key information concerning analytic small divisor problems in dimension 1. We show that T^p_α regularity, introduced by Calderón and Zygmund, is the only one which is relevant in order to unfold the pointwise regularity properties of B; we determine its T^p_α regularity at every point and show that it is directly related to the irrationality exponent $\tau(x)$: its p-exponent at x is exactly $1/\tau(x)$. This new example of multifractal function puts into light a new link between dynamical systems and fractal geometry. Finally we also determine the Hölder exponent of a primitive of B.

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1 Introduction

Let x be an irrational number in (0, 1), and let

$$x = [0; a_1, \ldots, a_n, \ldots] \tag{1}$$

denote its continued fraction expansion. The convergents p_n/q_n of x are

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

(following the standard tradition, we will not write explicitly the dependency of p_n, q_n and a_n in x except when it will be needed). The Brjuno function at x is

$$B(x) = \sum_{n=0}^{\infty} |p_{n-1} - q_{n-1}x| \log\left(\frac{p_{n-1} - x\, q_{n-1}}{q_n x - p_n}\right), \tag{2}$$

where, by convention,

$$(p_{-1}, q_{-1}) = (1, 0), \quad (p_0, q_0) = (0, 1), \quad \text{and} \quad (p_1, q_1) = (1, a_1),$$

so that the first term in (2) is $\log(1/x)$. The Brjuno function is extended by periodicity on $\mathbb{R} - \mathbb{Q}$.

The Brjuno function plays an important role in the theory of holomorphic dynamical systems: It was first introduced by Yoccoz [33], because of the information that it yields concerning analytic small divisor problems in dimension 1: Following Siegel [32], Brjuno [7,8] and Yoccoz [33], germs with linear part $e^{2i\pi x}$ are analytically conjugate to a rotation if and only if x is a *Brjuno number*, i.e. if $x \notin \mathbb{Q}$ and if the series defining $B(x)$ is convergent.

Marmi, Moussa and Yoccoz determined the optimal global regularity of B, showing that it belongs to *BMO*, see [22]. The *Marmi–Moussa–Yoccoz conjecture* is another regularity problem related with the Brjuno function: It states that the sum of B and the logarithm of the conformal radius of the Siegel disk of a monic quadratic polynomial is $C^{1/2}$, see [22, p. 267]. Key steps towards its resolution have been obtained by Buff et al. [9,14]. Local properties of B were recently investigated by Balazard and Martin in [3]: They showed that its Lebesgue points are precisely the Bruno numbers, and they obtained precise estimates of the average of B over an interval, which will play a key-role in our study, see e.g (9).

We will complement these regularity results by performing the *multifractal analysis* of the Brjuno function. The multifractal analysis of a function f usually consists into three steps:

- Choose a *pointwise regularity exponent* compatible with the global function space setting where f is considered,
- determine the value taken by this exponent at every point,
- compute the Hausdorff dimensions $\mathcal{D}_f(H)$ of the sets of points where this exponent takes a given value H.

The function $H \to \mathcal{D}_f(H)$ is the *multifractal spectrum* of f. Multifractal analysis has also been developped in the setting of measures and even of distributions, see e.g. [4,25,27] and references therein.

Several clues indicate that the tools supplied by multifractal analysis are relevant for the Brjuno function: First it is a *cocycle* under the action of $PGL(2, \mathbb{Z})$, as a consequence of the remarkable functional equations

$$\forall x \in \mathbb{R}\backslash\mathbb{Q}, \quad B(x+1) = B(x),$$
$$\forall x \in (0,1)\backslash\mathbb{Q}, \quad B(x) = \log(1/x) + xB(1/x),$$

see [23,24]. This property is reminiscent of the behavior of the Jacobi theta function under modular transforms, which is the key ingredient in the determination of the pointwise exponent of the *non-differentiable Riemann function* $\mathcal{R}(x) = \sum \sin(\pi n^2 x)/n^2$ [17], and of related trigonometric series [31]. Other trigonometric series also related to modular forms have been studied by Petrykiewicz in [28,29]. Finally, (2) also indicates that Diophantine approximation properties should play a role in the local regularity properties of B. This was the case for $\mathcal{R}$ [17], and several of its generalizations investigated by Chamizo et al. [11–13], and by Rivoal and Roques in [30]. Note that extremely few explicit deterministic functions playing an important role in mathematics have been proved to have a non-trivial multifractal spectrum: Most results in multifractal analysis are either of probabilistic or generic nature. Another motivation for performing such an analysis on B is that, beyond the important role played by this function, our result establishes a new relationship between holomorphic dynamical systems on one side, and real analysis and geometric measure theory on the other.

In order to perform the multifractal analysis of B, a first question is to determine a pointwise exponent fitted to its study. As mentioned above, this will be a consequence of the choice of a right function space setting. The two notions of pointwise regularity most commonly used are the *Hölder exponent*, defined for locally bounded functions and the *local dimension*, defined for positive Radon measures (see Sect. 4). However, these exponents are not fitted to the analysis of the Brjuno function for the following reasons. First, B is not locally bounded (i.e. does not coincide a.e. with a locally bounded function), because of the logarithmic singularities in (2) centered at all rational points (the series (2) is positive so that cancellations between terms cannot occur). As regards the local dimension, since B is positive, we can interpret it as the density of a positive Radon measure, but its local dimension is constant so that it is not adapted to measure the variations of regularity that exist in B. On other hand, these variations will be put into light through the use of a third notion of pointwise regularity, introduced by Calderón and Zygmund see [10], which is fitted to the study of functions that belong to L^p_{loc}.

Definition 1 Let $p \in [1, +\infty)$ and $\alpha \geq -1/p$. Let $f \in L^p_{loc}(\mathbb{R})$, and $x_0 \in \mathbb{R}$; f belongs to $T^p_\alpha(x_0)$ if there exist $C > 0$ and a polynomial P of degree less than α (with $P \equiv 0$ if $\alpha < 0$) such that, for ρ small enough,

$$\left(\frac{1}{2\rho} \int_{x_0-\rho}^{x_0+\rho} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq C\rho^\alpha. \tag{3}$$

The p-exponent of f at x_0 is

$$h^p_f(x_0) = \sup \left\{ \alpha : f \in T^p_\alpha(x_0) \right\}.$$

Let $p = 1$; if $f \in L^1_{loc}$, and if the left-hand side of (3) is a $o(1)$, then x_0 clearly is a Lebesgue point of f and the constant term of P is the Lebesgue value of f at x_0, i.e. is

$$\lim_{\rho \to 0} \frac{1}{2\rho} \int_{x_0-\rho}^{x_0+\rho} f(x)\, dx. \tag{4}$$

It follows that the 1-exponent measures the rate of convergence of the local averages (4) in the Lebesgue theorem. Therefore the determination of the 1-exponent that we will perform can be interpreted as a quantitative sharpening of the theorem of Balazard and Martin stating that every Brjuno point is a Lebesgue point of the Brjuno function. This is our main motivation for focusing on the case $p = 1$. However, in Sect. 4.2 we will deal with arbitrary ps (and conclude that, at any point, the p-exponent is independent of p). The 1-exponent of B at a point will be related with its *(Diophantine) irrationality exponent*.

Definition 2 Let $x_0 \notin \mathbb{Q}$, and p_n/q_n the sequence of convergents of the continued fraction expansion of x_0. Let $\tau_n(x_0)$ be defined by

$$\left| x_0 - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{\tau_n(x_0)}}.$$

The irrationality exponent (also called Diophantine approximation exponent or Diophantine order) of x_0 is

$$\tau(x_0) = \limsup_{n\to+\infty} \tau_n(x_0).$$

If x_0 is irrational, then $|x_0 - p_n/q_n| < 1/q_n^2$, so that $\tau_n(x_0) > 2$, and $\tau(x_0) \geq 2$. Let us recall the following equivalent definition for the irrationality exponent of x_0: $\tau(x_0)$ is the supremum of the $\tau \in \mathbb{R}$ such that there exists infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ such that $|x - p/q| \leq 1/q^\tau$.

Theorem 1 *If* $x_0 \in \mathbb{Q}$, *then* $h_B^1(x_0) = 0$. *Otherwise,*

$$h_B^1(x_0) = \frac{1}{\tau(x_0)}.$$

Remark Since almost every real number x satisfies $\tau(x) = 2$ it follows that h_B^1 takes the value $1/2$ almost everywhere. In the opposite direction, since quasi-every real number (in the sense of Baire categories) satisfies $\tau(x) = +\infty$, h_B^1 vanishes quasi-everywhere.

We now derive the consequence of Theorem 1 for multifractal analysis. Let $\dim(A)$ denote the Hausdorff dimension of the set A, with the convention $\dim(\emptyset) = -\infty$.

Definition 3 Let $p \in [1, +\infty)$ and $f \in L^p_{loc}(\mathbb{R})$. The level sets of h_f^p, denoted by E_H^p, are

$$\forall H \in \left[-\frac{1}{p}, +\infty\right], \quad E_H^p = \left\{x : h_f^p(x) = H\right\}.$$

The p-spectrum of f is the function $\mathcal{D}_f^p : [-1/p, +\infty] \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\mathcal{D}_f^p(H) = \dim\left(E_H^p\right).$$

Few p-spectrums have been determined: Let us mention the characteristic functions of some fractal sets [20] and random wavelet series [1]; generic results (in the Baire and prevalence settings) for functions in a Sobolev space were obtained by Fraysse [15]; recently, 2-exponents of trigonometric series which are not locally bounded were obtained by Seuret and Ubis [31].

The precised formulation of Jarnik's theorem states that

$$\dim\{x : \tau(x) = t\} = \frac{2}{t}, \tag{5}$$

see e.g. [17]. Therefore the 1-spectrum of B follows from Theorem 1.

Corollary 1 *The* 1*-spectrum of B is*

$$\mathcal{D}^1_B(H) = \begin{cases} 2H & \textit{if} \quad H \in [0, 1/2], \\ -\infty & \textit{else.} \end{cases} \tag{6}$$

Remark Since (5) also holds after restricting to the points x inside a nonempty open interval, it follows that the multifractal spectrum of B restricted to any interval (a, b) of positive length is also given by (6). Following [5], B is an *homogeneous multifractal function*.

Theorem 1 is proved in Sect. 2. The computation of the 1-exponent is sharpened in Sect. 3 where the exact modulus of continuity of B at *badly approximable numbers* is determined. Results concerning other notions of pointwise regularity are grouped in Sect. 4. Finally, we mention related open problems in Sect. 5.

2 Determination of the 1-exponent of *B*

The fact that $B \in BMO$ implies a uniform lower bound on the 1-exponent. Indeed it follows from the John–Nirenberg inequality (or from Proposition 3 of [3]) that

$$\exists C > 0, \ \forall x, y : \quad |x - y| \le \frac{1}{2}, \quad \left| \int_x^y B(t)\, dt \right| \le C|x - y| \log \left(\frac{1}{|x - y|} \right) \tag{7}$$

(here and in the following, the value of the constant C may change from one line to the next). Thus, for $h < 1/2$,

$$\forall D, \quad \frac{1}{2h} \int_{x_0 - h}^{x_0 + h} |B(x) - D| dx \le C \log(1/h)$$

and finally,

$$\forall x_0, \quad h^1_B(x_0) \ge 0. \tag{8}$$

Following [3], it will be convenient to define a function $\tilde{B}$ at rationals in the following way: If $x_0 \in (0, 1) \cap \mathbb{Q}$, then the continued fraction expansion (1) of x_0 stops at a rank N, and

$$\tilde{B}(x_0) = \sum_{n=0}^{N-1} |p_{n-1} - q_{n-1} x_0| \ \log \left(\frac{p_{n-1} - x_0\, q_{n-1}}{q_n x_0 - p_n} \right);$$

for instance, for $N = 1$, $\tilde{B}(1/k) = \log k$.

The regularity of B at rationals is a consequence of the following estimate of Balazard and Martin (Proposition 12 of [3]): Let $r = p/q$ with $p \wedge q = 1$;

$$\text{if } |h| < \frac{2}{3q^2}, \ \frac{1}{h}\int_r^{r+h} B(x)dx = \frac{\log(e/q^2|h|)}{q} + \tilde{B}(r) + \mathcal{O}\left(qh\log\left(\frac{1}{q^2|h|}\right)\right) \tag{9}$$

where the $\mathcal{O}$ is uniform (in p, q and h). In particular, if $x_0 = p/q$ is rational, then $\forall D$, for h small enough,

$$\int_{x_0-h}^{x_0+h} |B(x) - D|dx \geq \frac{h}{2q}\log(1/h), \tag{10}$$

so that, at rationals $h^1_B(x_0) = 0$. More precisely, by (7), for C large enough, the function $Ch\log(1/h)$ is a uniform 1-modulus of continuity of B (which will be defined further at Definition 4); and, up to the multiplicative constant, this is optimal, because it follows from (7) and (10) that $h\log(1/h)$ is the order of magnitude of the left hand side of (10).

The regularity of B at Cremer numbers (i.e. at irrationals that are not Brjuno numbers) follows from the fact that they are not Lebesgue points, see Proposition 14 of [3]. Thus, it follows from (4) that $h^1_B(x_0) \leq 0$ and, using (8), $h^1_B(x_0) = 0$. Therefore, from now on, we can assume that x_0 is a Brjuno number, so that $B(x_0)$ is finite, and its values (pointwise and in the Lebesgue sense) coincide.

2.1 Global and pointwise irregularity of the Brjuno function

The idea for proving the irregularity of B at Brjuno numbers is to reinterpret (9) as implying that some of its wavelet coefficients are large in the neighbourhood of the point considered, so that B is irregular at those points. We will need a variant of the classical wavelet criterium (such as in [16]).

We assume in the following that ψ is a bounded, compactly supported function satisfying

$$\sup_{x\in\mathbb{R}} |\psi(x)| \leq 1 \quad \text{and} \quad \int_{\mathbb{R}} \psi(x)dx = 0; \tag{11}$$

such a function ψ will be called an *admissible wavelet*. Let

$$\forall a > 0, \quad b \in \mathbb{R}, \quad \psi_{a,b}(x) = \psi\left(\frac{x-b}{a}\right).$$

If $f \in L^1_{loc}(\mathbb{R})$, the continuous wavelet transform of f is

$$C_f(a,b) = \frac{1}{a}\int_{\mathbb{R}} f(x)\psi_{a,b}(x)dx.$$

In order to obtain sharp results, we need to extend the notion of T^p_α regularity to general moduli of continuity. We start by defining the possible candidates: A function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies hypothesis $\mathcal{H}$ if

$$(\mathcal{H}) \quad \begin{cases} \theta(0) = 0, \\ \theta \text{ is continuous and non-decreasing in a neighborhood of } 0. \end{cases}$$

Definition 4 Let θ be a function satisfying $\mathcal{H}$ and $f \in L^p_{loc}(\mathbb{R})$; θ is a p-modulus of continuity of f at x_0 if there exists a polynomial P such that, for ρ small enough,

$$\left(\int_{x_0-\rho}^{x_0+\rho} |f(x) - P(x-x_0)|^p dx\right)^{1/p} \leq \theta(\rho). \tag{12}$$

Note that T^p_α regularity corresponds to $\theta(\rho) = C\rho^{\alpha+1/p}$.

Lemma 1 *Let ψ be an admissible wavelet, $p \in [1,+\infty]$ and $f \in L^p_{loc}(\mathbb{R})$; let θ be a p-modulus of continuity of f at x_0 satisfying*

$$\exists C > 0, \ \forall \rho \in (0,1], \quad \theta(\rho) \geq C\rho^{1+1/p}.$$

Then

$$supp\ (\psi_{a,b}) \subset [x_0-\rho, x_0+\rho] \implies |C_f(a,b)| \leq 2^{1-1/p}\ \theta(\rho)\frac{\rho^{1-1/p}}{a}.$$

This result will be used for $p = 1$ in order to prove the pointwise irregularity of the Brjuno function.

Proof The growth condition on θ implies that we can restrict to polynomials P of degree 0. Since ψ has a vanishing first moment,

$$\forall D \in \mathbb{R}, \quad C_f(a,b) = \frac{1}{a}\int_{\mathbb{R}} (f(x) - D)\psi_{a,b}(x)dx.$$

Using (11), we get

$$|C_f(a,b)| \leq \frac{1}{a}\int_{x_0-\rho}^{x_0+\rho} |f(x) - D|dx \leq 2^{1-1/p}\ \frac{\theta(\rho)}{a}\rho^{1-1/p}. \qquad \square$$

Applying (9) to h and $h/2$, we obtain that, if $0 < h < 2/3q^2$, then

$$\frac{1}{h}\int B(x)H\left(\frac{x-r}{h}\right)dx = \frac{\log 2}{q} + \mathcal{O}\left(qh\log\left(\frac{1}{q^2h}\right)\right), \tag{13}$$

where $H = 1_{[0,1/2]} - 1_{[1/2,1]}$ is the Haar wavelet. Hence the following result holds.

Lemma 2 *Let $r = p/q$ with $p \wedge q = 1$. If $0 < h < 2/3q^2$, then*

$$\left|C_B\left(h, \frac{p}{q}\right) - \frac{\log 2}{q}\right| \leq \tilde{C}qh\log\left(\frac{1}{q^2h}\right), \tag{14}$$

where the wavelet used is the Haar wavelet and the constant $\tilde{C}$ is independent of p, q and h.

We now introduce a notion of uniform irregularity associated with moduli of continuity for $p = 1$.

Definition 5 Let θ be a function satisfying $\mathcal{H}$. A function $f \in L^1_{loc}(\mathbb{R})$ is uniformly θ-irregular if

$$\forall x_0,\ \forall P,\ \exists \rho_n \to 0: \quad \int_{x_0-\rho_n}^{x_0+\rho_n} |f(x) - P(x-x_0)|\,dx \geq \theta(\rho_n).$$

Proposition 1 *There exists $A > 0$ such that B is uniformly θ-irregular for $\theta(\rho) = A\rho^{3/2}$; and this result is optimal* (*i.e.* $\theta(\rho)$ *cannot be replaced by a* $o(\rho^{3/2})$).

The optimality of Proposition 1 will be proved in Sect. 3 by considering badly approximable numbers.

Proof Let $x_0 \in \mathbb{R}$. First note that, if $x_0 \in \mathbb{Q}$, then the result follows from (10). If $x_0 \notin \mathbb{Q}$, we apply (14) to the sequence

$$r_n = \frac{p_n}{q_n}$$

of convergents of x_0. We now pick $h_n = \varepsilon/q_n^2$, where ε is positive and such that $\tilde{C}\varepsilon\log(1/\varepsilon) \leq 1/4$ (where $\tilde{C}$ is the constant in Lemma 2). It follows that

$$C_B(h_n, r_n) \geq \frac{1}{4q_n}.$$

We now apply Lemma 1 with $a = h_n, b = r_n$ and $\rho_n = |x_0 - r_n| + h_n$; if θ is a 1-modulus of continuity at x_0, then

$$|C_B(h_n, r_n)| \leq \frac{\theta(\rho_n)}{h_n},$$

which implies that

$$\frac{1}{4q_n} \leq \frac{\theta(\rho_n)}{h_n}.$$

Using that $\rho_n \leq 2/q_n^2$ and θ is increasing, if follows that

$$\frac{\varepsilon}{4q_n^3} \leq \theta(\rho_n) \leq \theta(2/q_n^2) = \frac{2A\sqrt{2}}{q_n^3},$$

hence a contradiction if A is small enough. □

Proposition 1 implies that the 1-exponent satisfies $\forall x \in \mathbb{R}, h_B^1(x) \leq 1/2$; thus we can assume in the following that the polynomial in (12) boils down to a constant which has to be $B(x_0)$ as x_0 is a Brjuno number (recall that Brjuno numbers are Lebesgue points).

Let us now check that the same argument as in the proof of Proposition 1 yields an irregularity result at points x_0 for which $\tau(x_0) > 2$. Recall that an irrational point x_0 is τ-*well approximable* if

$$|x_0 - r_n| \leq \frac{1}{q_n^\tau},$$

for infinitely many ns.

Lemma 3 *Let $\tau > 2$. If x_0 is τ-well approximable, then $\theta(\rho) = \frac{1}{8}\rho^{1+1/\tau}$ is not a modulus of continuity of B at x_0 (so that $h_B^1(x_0) \leq 1/\tau$).*

Proof Assume that $\theta(\rho) = \frac{1}{8}\rho^{1+1/\tau}$ is a modulus of continuity of B at x_0. We pick $h_n = 1/q_n^\tau$. As above $C_B(1/q_n^\tau, r_n) \sim \log(2)/q_n$ when $n \to +\infty$. We apply Lemma 1 with $a = h_n, b = r_n$ and $\rho_n = |x_0 - r_n| + h_n$ so that $\rho_n \leq 2/q_n^\tau$. We get $1/2 \leq 2^{1+1/\tau}/8$, hence a contradiction. □

Recall that an irrational number x_0 is Diophantine if $\tau(x_0) < \infty$; Liouville numbers are the irrational numbers that are not Diophantine. It follows from Lemma 3 that *the* 1*-exponent of the Brjuno function vanishes at Liouville numbers.* Moreover if x_0 is such that $\tau(x_0) > 2$, then Lemma 3 gives

$$h_B^1(x_0) \leq \frac{1}{\tau(x_0)}.$$

2.2 Pointwise regularity of the Brjuno function

We now prove regularity for B at Diophantine numbers of $X = (0, 1)\backslash\mathbb{Q}$.

We begin by recalling basic points about the continued fraction expansion of irrational numbers. First, the Gauss map $A\colon X \to X$ is defined by

$$A(x) = \left\{\frac{1}{x}\right\}$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x and $\lfloor x \rfloor$ its integer part of x. For $n \in \mathbb{N}$, we denote by A^n the nth iterate of A.

If $x \in X$ and $n \geq 0$, $\quad A^n(x) = \dfrac{p_{n-1}(x) - x\, q_{n-1}(x)}{q_n(x)x - p_n(x)}$ and $\lfloor A^n(x) \rfloor = a_n(x)$,

see e.g. [6, pp. 40–41].

We will denote by $\mathfrak{c}[b_1, \ldots, b_k]$ the open sub-interval of (0, 1) with endpoints $[0; b_1, \ldots, b_k]$ and $[0; b_1, \ldots, b_{k-1}, b_k + 1]$. These intervals are called *cylinders of order* k. Note that in a cylinder of order k, A^k is continuous, and for all $j \leq k$ the functions a_j, p_j and q_j are constant. For $x \in X$, let

$$\beta_n(x) = |x q_n(x) - p_n(x)|$$

and

$$\gamma_n(x) = \beta_{n-1}(x) \log\left(\frac{1}{A^n(x)}\right)$$

so that

$$B(x) = \sum_{n=0}^{\infty} \gamma_n(x).$$

We recall the well-known bounds (see e.g [6, p. 42])

$$\frac{1}{2q_{n+1}(x)} \leq \beta_n(x) \leq \frac{1}{q_{n+1}(x)}. \tag{15}$$

It follows that

$$q_{n+1}(x) \leq q_n(x)^{\tau_n(x)-1}. \tag{16}$$

Let us also recall (see Proposition 1 of [3]) that for $k \geq 1$,

$$\frac{\log(q_{k+1}(x))}{q_k(x)} - \frac{\log(2q_k(x))}{q_k(x)} \leq \gamma_k(x) \leq \frac{\log(q_{k+1}(x))}{q_k(x)}. \tag{17}$$

Let $x_0 \in X$. In the sequel, a_k, p_k, q_k denote the value at x_0 of the functions a_k, p_k, q_k. We need to estimate the integrals

$$\int_I \gamma_k(t)\, dt,$$

where

$$I = (x_0 - \rho/2; x_0 + \rho/2) \quad \text{with} \quad \rho > 0. \tag{18}$$

These estimates will depend on an integer K which is defined as follows: $K\ (= K(I))$ is the largest integer such that

$$I \subseteq \mathfrak{c}[a_1, \ldots, a_K]. \tag{19}$$

We also denote by $\{F_k\}_{k\geq 0}$ the sequence of Fibonacci numbers (i.e. $F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$).

Lemma 4 *Let $x_0 \in X$, let ρ be such that $0 < \rho < e^{-2}$ with $x_0 - \rho/2$ and $x_0 + \rho/2$ irrational. Let I be the interval be given by (18) and K the integer defined by (19). There exists an absolute constant $C > 0$ such that for $K \geq 1$,*

$$\forall k < K, \quad \int_I \left|\gamma_k(x) - \gamma_k(x_0)\right| dx \leq C q_{k+1} \rho^2, \tag{20}$$

$$\int_I \left|\gamma_K(x) - \gamma_K(x_0)\right| dx \leq C q_{K+1} \rho^2 \log(q_{K+1}), \tag{21}$$

$$\forall k > K, \quad \int_I \gamma_k(x) dx \leq C \frac{\rho}{F_{k-K}} \left(\frac{\log(1/\rho)}{q_{K+1}} + \rho^{1/2}\right). \tag{22}$$

Proof The bound (20) is exactly Proposition 7 of [3]. Propositions 9 and 10 of the same paper give bounds for the integrals $\int_I \gamma_k(x) dx$ for $k \geq K$, however (21) and (22) cannot be directly derived from them, but will be a consequence of their proofs. Let us recall the notations of [3]: We set

$$x_0 + (-1)^K \rho/2 = \frac{u p_K + p_{K-1}}{u q_K + q_{K-1}} \quad \text{and}$$
$$x_0 + (-1)^{K-1} \rho/2 = \frac{v p_K + p_{K-1}}{v q_K + q_{K-1}},$$

and

$$m = [u] \quad \text{and} \quad n = [v], \tag{23}$$

so that $1 \leq m \leq a_{K+1} \leq n$. By maximality of K, we have $n > m$. Inequality (40) of [3] gives

$$\rho \geq \frac{v - u}{6 q_K^2 m n}. \tag{24}$$

Let us now prove (21). We distinguish two cases.

First, suppose that $n \geq 2m+1$. Then $v-u \geq (n-m)/2$ and we obtain from (24) that

$$\rho \geq \frac{1}{24mq_K^2} \geq \frac{1}{24a_{K+1}q_K^2} \geq \frac{1}{24q_K q_{K+1}}.$$

Proposition 9 of [3] and (17) give

$$\int_I \gamma_K(x) \leq C\rho \frac{\log(q_{K+1})}{q_K}, \tag{25}$$

from which we deduce

$$\int_I \left|\gamma_K(x) - \gamma_K(x_0)\right| dx \leq C\frac{\rho}{q_K}\log(q_{K+1}) \leq C\rho^2 q_{K+1}\log(q_{K+1}).$$

Suppose now that $m \leq n \leq 2m$. If $x \in I$, the derivative of γ_K satisfies

$$\gamma_K'(x) = (-1)^{K-1}\Big(q_{K-1}(x)\log(1/A^K(x)) + \beta_K(x)^{-1}\Big),$$

so that

$$\left|\gamma_K'(x)\right| \leq Cq_{K+1}(x).$$

For $x \in I$, there exists $m \leq \ell \leq n$ such that $x \in \mathfrak{c}[a_1, \ldots, a_K, \ell]$ which yields

$$q_{K+1}(x) = \ell q_K + q_{K-1} \leq nq_K + q_{K-1} \leq 2a_{K+1}q_K + q_{K-1} \leq 2q_{K+1}.$$

By the mean-value theorem,

$$\int_I \left|\gamma_K(x) - \gamma_K(x_0)\right| dx \leq Cq_{K+1}\rho^2,$$

and the case $k = K$ is settled.

It remains to consider the case $k > K$. Let

$$E = n - m + 1. \tag{26}$$

If $E = 2$, inequality (43) of [3] gives

$$\int_I \gamma_k(x)dx \leq \frac{2e}{q_{K+1}F_{k-K}}\rho\log(1/\rho). \tag{27}$$

If $E \geq 3$, then $v - u \geq (n-m)/2$ so that (24) gives $\rho \geq \frac{n-m}{12 q_K^2 mn}$. Using

$$\int_I \gamma_k(x)dx \leq 6 \frac{n-m}{q_K^3 F_{k-K} m^2 n} \tag{28}$$

(see p. 213 of [3]), it follows that

$$\int_I \gamma_k(x)dx \leq \frac{(12\rho)^{3/2}}{F_{k-K}} \Big(\frac{n}{(n-m)m} \Big)^{1/2} \leq \frac{(12\rho)^{3/2}}{F_{k-K}}.$$

□

Proposition 2 *Let x_0 be a Diophantine number, and $\varepsilon > 0$. There exists $C = C(x_0) > 0$ and $\rho_0 = \rho_0(x_0, \varepsilon) > 0$ such that, if $0 < \rho < \rho_0$, then*

$$\frac{1}{\rho} \int_{x_0 - \rho/2}^{x_0 + \rho/2} |B(x_0) - B(x)| dx \leq C \rho^{1/(\tau(x_0)+\varepsilon)} \log(1/\rho). \tag{29}$$

From (29) we deduce that, if x_0 is Diophantine, then $h_B^1(x_0) \geq \frac{1}{\tau(x_0)}$ which ends the proof of Theorem 1.

Proof We may assume that $x_0 \pm \rho/2$ are both irrational. Let $\varepsilon > 0$. There exists an integer $K_0 = K_0(x_0, \varepsilon)$ such that

$$\forall K \geq K_0, \quad \tau_K(x_0) \leq \tau(x_0) + \varepsilon. \tag{30}$$

We will note $\tau_k(x_0) = \tau_k$ and $\tau(x_0) = \tau$. Following [3], $\delta_k = \delta_k(x_0)$ will denote the distance from x_0 to the endpoints of $\mathfrak{c}[a_1, \ldots, a_k]$, i.e.

$$\delta_k = \min \Big(\Big| x_0 - \frac{p_k}{q_k} \Big|, \Big| x_0 - \frac{p_k + p_{k-1}}{q_k + q_{k-1}} \Big| \Big).$$

Proposition 4 of [3] gives

$$\delta_k \leq \frac{1}{q_k q_{k+1}} \quad \text{and} \quad \delta_k \geq \begin{cases} \frac{1}{2 q_{k+1} q_{k+2}} & \text{if} \quad a_{k+1} = 1, \\ \frac{1}{2 q_k q_{k+1}} & \text{if} \quad a_{k+1} \geq 2. \end{cases} \tag{31}$$

Let $K = K(x_0, \rho)$ be the largest integer such that $I = (x_0 - \rho/2; x_0 + \rho/2)$ is included in $\mathfrak{c}[a_1, \ldots, a_K]$. We have

$$\frac{1}{2 q_{K+2} q_{K+3}} \leq \delta_{K+1} < \rho/2$$

so that $K \to +\infty$ when $\rho \to 0$. Let $0 < \rho_0 < e^{-2}$ be such that for all $0 < \rho < \rho_0$, $K \geq \max(K_0, 1)$ and let us evaluate for $\rho < \rho_0$,

$$\int_I |B(x_0) - B(x)|dx \leq \sum_{k \leq K} \int_I |\gamma_k(x_0) - \gamma_k(x)|dx + \rho \sum_{k>K} \gamma_k(x_0) + \sum_{k>K} \int_I \gamma_k(x)dx.$$

Using (20) and (21), and since the sequence $\{q_k\}_{k\geq 0}$ grows (at least) exponentially, it follows that

$$\sum_{k\leq K} \int_I |\gamma_k(x_0) - \gamma_k(x)|dx \leq C\rho^2 q_{K+1} \log(q_{K+1}).$$

Using (22), we get

$$\sum_{k>K} \int_I \gamma_k(x)dx \leq C\rho\Big(\frac{\log(1/\rho)}{q_{K+1}} + \rho^{1/2}\Big).$$

Since x_0 is Diophantine, $\tau(x_0) < \infty$; therefore the sequence $(\tau_k)_{k\geq 0}$ is bounded. Using (16) and (17), we get for $k > K$, $|\gamma_k(x_0)| \leq C \log(q_k)/q_k$ where C depends on x_0. Thus

$$\sum_{k>K} \gamma_k(x_0) \leq C \frac{\log(q_{K+1})}{q_{K+1}}.$$

Collecting these estimates we get

$$\int_I |B(x_0) - B(x)|dx \leq C\rho\Big(\rho\, q_{K+1} \log(q_{K+1}) + \frac{\log(1/\rho)}{q_{K+1}} + \rho^{1/2}\Big). \tag{32}$$

According to (16)

$$q_{K+1} \leq q_K^{\tau_K - 1} = \left|x - \frac{p_K}{q_K}\right|^{(1-\tau_K)/\tau_K} \leq \rho^{-1+1/\tau_K},$$

from which we deduce that $\log(q_{K+1}) \leq C \log(1/\rho)$. If $a_{K+2} \geq 2$, according to (15) and (31),

$$\frac{1}{q_{K+1}^{\tau_{K+1}}} = \left|x - \frac{p_{K+1}}{q_{K+1}}\right| \leq \frac{1}{q_{K+1}q_{K+2}} \leq 2\delta_{K+1} < \rho,$$

and if $a_{K+2} = 1$ we get in the same way

$$\frac{1}{q_{K+1}} \le \frac{2}{q_{K+2}} \le 2\rho^{1/\tau_{K+2}}.$$

Inserting these estimates in (32), we finally get

$$\int_I |B(x_0) - B(x)|dx \le C\rho\left(\rho^{1/\tau_K} + \rho^{1/\tau_{K+1}} + \rho^{1/\tau_{K+2}}\right)\log(1/\rho),$$

and the conclusion follows from (30). □

3 Badly approximable numbers

Theorem 1 can be interpreted as stating that the slower the sequence q_n increases, the smoother B is at x_0. We now prove that, indeed, the points for which the sequence q_n grows as slowly as possible are the ones where B is the smoothest.

An irrational number x_0 is *badly approximable* if the sequence of $\{a_k\}_{k\ge 0}$ is bounded, or, equivalently, if

$$\exists C > 0, \quad \forall p, q \ne 0, \quad \left|x_0 - \frac{p}{q}\right| \ge \frac{C}{q^2}.$$

It follows that $\tau(x_0) = 2$; thus we already know that $h^1_B(x_0) = 1/2$. We now sharpen this result.

Proposition 3 *A point $x_0 \in (0, 1)$ is badly approximable if and only if there exists $C > 0$ such that $\theta(\rho) = C\rho^{3/2}$ is a modulus of continuity of B at x_0.*

A consequence is the optimality of Proposition 1 (up to the multiplicative constant): Badly approximable numbers have the smallest possible modulus of continuity.

Proof First note that a function which is a $o(\rho^{3/2})$ cannot be a modulus of continuity at badly approximable numbers, as a consequence of Proposition 1. We now prove that, for C large enough, $C\rho^{3/2}$ is a modulus of continuity at such a number. In this proof, the values of C may change from one line to the next, but only depend on x_0. We will use the same notations (I, K, E, δ_K) as in the proofs of Lemma 4 and Proposition 2. First note that

$$\rho \le \left|x_0 - \frac{p_K}{q_K}\right| \le \frac{1}{q_K^2}, \tag{33}$$

and also, as x_0 is badly approximable,

$$\exists C : \frac{\rho}{2} > \delta_{K+1} \geq \frac{1}{2q_{K+2}q_{K+3}} \geq \frac{C}{q_K^2}. \tag{34}$$

According to (20) and (33),

$$\sum_{k<K} \int_I |\gamma_k(x) - \gamma_k(x_0)|dx \leq C\rho^2 q_K \leq C\rho^{3/2}. \tag{35}$$

Let $k > K$. The proof of Proposition 10 of [3, p. 213] contains the following inequality for $E \geq 3$, which actually remains true for $E \geq 2$:

$$\int_I \gamma_k(t)dt \leq \frac{2}{q_K^3 F_{k-K}} \sum_{m\leq \ell \leq n} \frac{1}{\ell^3}.$$

This inequality and (34) directly imply

$$\sum_{k>K} \int_I \gamma_k(x)dx \leq C\rho^{3/2}. \tag{36}$$

As $1/A^k(x_0) = a_{k+1}(x_0) + A^{k+1}(x_0)$, there exists $C > 0$ such that for all $k \in \mathbb{N}$, $\log(1/A^k(x_0)) \leq C$; using the exponential growth of the $(q_k)_{k\geq 0}$, we get

$$\rho \sum_{k\geq K} \gamma_k(x_0) \leq C\rho \sum_{k\geq K} \frac{1}{q_k} \leq C\frac{\rho}{q_K} \leq C\rho^{3/2}. \tag{37}$$

To treat $\int_I \gamma_K(x)dx$, we use (39) of [3]:

$$\int_I \gamma_K(x)dx \leq \frac{1}{q_K^3} \int_{A^K(I)} \log(1/u)du \leq \frac{1}{q_K^3} \int_0^{|A^K(I)|} \log(1/u)du \leq C\rho^{3/2}, \tag{38}$$

for $u \mapsto \log(1/u)$ is decreasing on $(0, 1]$ and $\int_0^1 \log(1/u)du < \infty$. Collecting the estimates (35), (36), (37) and (38) we obtain that $C\rho^{3/2}$ is a modulus of continuity at x_0.

We now prove that badly approximable points are the only ones for which the modulus of continuity is equivalent to $\rho^{3/2}$. Let $h_n = |x_0 - p_n/q_n|$; x_0 is not badly approximable if and only if there exists a subsequence $n(m)$ such that for $m \to \infty$,

$$h_{n(m)} = o\left(\frac{1}{(q_{n(m)})^2}\right). \tag{39}$$

The proof then follows the one of Proposition 1: On one hand, (14) implies that

$$C_B\left(h_{n(m)}, \frac{p_{n(m)}}{q_{n(m)}}\right) \geq \frac{1}{4q_{n(m)}};$$

on other hand, applying Lemma 1 with $\rho_{n(m)} = 2h_{n(m)}$, we obtain that, if $C\rho^{3/2}$ is a modulus of continuity at x_0, then

$$\frac{1}{4q_{n(m)}} \leq \frac{C(\rho_{n(m)})^{3/2}}{h_{n(m)}} \leq C2^{3/2}(h_{n(m)})^{1/2},$$

which contradicts (39). □

4 Additional pointwise regularity results

We start by showing why the pointwise exponent used for positive measures is not relevant for B. Recall that, if μ is a positive Radon measure defined on $\mathbb{R}$, The *local dimension* of μ at x_0 is

$$\underline{\dim}_{loc}(\mu, x_0) = \liminf_{\rho\to 0^+} \frac{\log \mu([x_0-\rho, x_0+\rho])}{\log\, \rho}.$$

The local dimension is well defined for the Brjuno function; however, it does not allow to capture possible changes in its pointwise regularity. Indeed, let us check that it is constant.

First, clearly, $\exists C > 0$ such that $\forall x \in \mathbb{R}$, $B(x) \geq C$ because there is no cancellation in the series (2); so that $\forall x$, $\underline{\dim}_{loc}(\mu, x) \leq 1$. On other hand, since $B \in BMO$, it follows immediately from (7) that $\forall x$, $\underline{\dim}_{loc}(\mu, x) \geq 1$.

The p-exponent is better fitted to measure variations of regularity of B because its definition allows for the substraction of an appropriate polynomial.

4.1 Hölder regularity of the primitive of B

The proof of Theorem 1 strongly uses (9), which estimates increments of the primitive of B; therefore a natural question is to wonder if it can yield its *Hölder exponent*.

Definition 6 Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $x_0 \in \mathbb{R}$ and $\alpha \geq 0$. The function f belongs to $C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P of degree less than α such that, for ρ small enough,

$$\sup \text{ess}_{|x-x_0|\leq\rho} |f(x) - P(x-x_0)| \leq C\rho^\alpha. \qquad (40)$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha \geq 0 \mid f \in C^\alpha(x_0)\}.$$

We denote by $\mathfrak{B}$ a primitive of B. A lower bound for $h_{\mathfrak{B}}$ is a consequence of the following (straightforward) remark: Let $f \in L^1_{loc}(\mathbb{R})$, and let F denote a primitive of f; then

$$\forall x_0 \in \mathbb{R}, \quad h_F(x_0) \geq h^1_f(x_0) + 1. \tag{41}$$

In general, equality does not hold in (41); we will now check that it holds everywhere in the case of the Brjuno function.

Proposition 4 *If $x_0 \in \mathbb{Q}$, then $h_{\mathfrak{B}}(x_0) = 1$. Otherwise,*

$$h_{\mathfrak{B}}(x_0) = 1 + \frac{1}{\tau(x_0)}.$$

In order to prove this result, we will need an irregularity criterium based on finite differences. We note

$$\Delta_2 f(x,h) = 2f\left(x + \frac{h}{2}\right) - f(x+h) - f(x). \tag{42}$$

Lemma 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function; let $\alpha < 2$ and $\gamma \geq 0$. Let $x_0 \in \mathbb{R}$, and assume that there exist $\rho_n > 0$, h_n, and r_n such that*

$$r_n \in [x_0 - \rho_n, x_0 + \rho_n], \quad \rho_n \to 0, \quad \textit{and} \quad \frac{\rho_n}{|\log \rho_n|^\gamma} \leq |h_n| \leq \rho_n.$$
$$\textit{If} \quad |\Delta_2 f(r_n, h_n)| \geq |h_n|^\alpha, \quad \textit{then} \quad h_f(x_0) \leq \alpha.$$

Proof Clearly, if f is continuous, then the sup ess in (40) can be replaced by a sup. Therefore, if $f \in C^\beta(x_0)$ for a $\beta < 2$, then there exists a polynomial P of degree at most 1 and $r > 0$ such that

$$\forall x \in [x_0 - r, x_0 + r], \quad f(x) = P(x - x_0) + O(|x - x_0|^\beta). \tag{43}$$

Using (43) for $x = r_n,\ r_n + h_n/2$ and $r_n + h_n$ in (42), we get

$$\Delta_2 f(r_n, h_n) = O(\rho_n^\beta) = O\big(|h_n|^\beta \big|\log |h_n|\big|^{\beta\gamma}\big).$$

Therefore, if $|\Delta_2 f(r_n, h_n)| \geq |h_n|^\alpha$, then $\forall \beta > \alpha$, $f \notin C^\beta(x_0)$. □

Let us now prove Proposition 4. The case $x_0 \in \mathbb{Q}$ follows from (9). If $x_0 \notin \mathbb{Q}$, (41) and Theorem 1 imply that $h_{\mathfrak{B}}(x_0) \geq 1 + 1/\tau(x_0)$. Note that (13) can be rewritten

$$\text{If} \quad |h| < \frac{2}{3q^2}, \quad \text{then} \quad \frac{1}{h}\Delta_2\mathfrak{B}(r,h) = \frac{\log 2}{q} + \mathcal{O}\left(qh\log\left(\frac{1}{q^2|h|}\right)\right).$$

Let now $\tau \geq 2$ and assume that x_0 is τ-well approximable. We can assume, by extracting a subsequence if necessary, that for all $n \geq 1$, $\left|x_0 - p_n/q_n\right| \leq 1/q_n^\tau$, and we pick for r the sequence of convergents $r_n = p_n/q_n$, $\rho_n = 1/q_n^\tau$, and $h_n = 1/q_n^\tau(\log q_n)^2$. We obtain that

$$\Delta_2\mathfrak{B}(r_n, h_n) = \frac{\log 2}{\tau^{2/\tau}}\,(h_n)^{1+1/\tau}|\log(h_n)|^{2/\tau}\Big(1 + o(1)\Big),$$

and Lemma 5 implies that $h_{\mathfrak{B}}(x_0) \leq 1 + 1/\tau$. □

4.2 The *p*-exponent of *B* for *p* > 1

Theorem 1 can be extended to p-exponents in the following way:

$$\forall p \geq 1,\ \forall x_0 \in \mathbb{R}, \quad h_B^p(x_0) = h_B(x_0). \tag{44}$$

We outline the proof of this result. Using the John–Nirenberg inequality,

$$\exists C > 0, \forall x, y: \quad |x - y| \leq \frac{1}{2}, \quad \left|\int_x^y B(t)^p\, dt\right|^{1/p} \leq C|x - y|^{1/p}\log\left(\frac{1}{|x - y|}\right),$$

so that $\forall x_0 \in (0,1)$, $h_B^p(x_0) \geq 0$. On other hand, Hölder's inequality implies that

$$h_B^p(x_0) \leq h_B^1(x_0). \tag{45}$$

Hence it follows from Theorem 1 that if x_0 is a rational or a Liouville number, then $h_B^p(x_0) = 0$.

Suppose now that x_0 is Diophantine. The results obtained in Sect. 2.2 are based on the estimates (20)–(28) from [3]. They extend as follows: Let ρ be such that $0 < \rho < e^{-2}$ with $x_0 - \rho/2$ and $x_0 + \rho/2$ irrational, $I = (x_0 - \rho/2; x_0 + \rho/2)$, and K, m, n, E the integers defined by (19), (23) and (26). There exists an absolute constant $C > 0$ such that for $K \geq 1$,

$$\forall k < K, \quad \left(\int_I \left|\gamma_k(x) - \gamma_k(x_0)\right|^p dx\right)^{1/p} \leq Cq_{k+1}\rho^{1+1/p}, \tag{46}$$

$$\left(\int_I \gamma_K(x)^p dx\right)^{1/p} \le C\,\frac{\rho^{1/p}\log(q_{K+1})}{q_K}, \tag{47}$$

$$\forall k > K,\quad \left(\int_I \gamma_k(x)^p dx\right)^{1/p} \le C\,\frac{\rho^{1/p}\log(1/\rho)}{F_{k-K}q_{K+1}},\ \text{if } E = 2, \tag{48}$$

$$\left(\int_I \gamma_k(x)^p dx\right)^{1/p} \le C\,\frac{(n-m)^{1/p}}{F_{k-K}q_K^{1+2/p}m^{1+1/p}n^{1/p}}\ \text{if } E \ge 3. \tag{49}$$

The proofs follow the same arguments as in the proofs of Propositions 7, 9, 10 of [3].

Starting from (46)–(49), one obtains the following extension of Lemma 4: Under the same hypothesis there exists an absolute constant $C > 0$ such that for $p \ge 1$, $K \ge 1$,

$$\forall k < K,\quad \left(\int_I \left|\gamma_k(x) - \gamma_k(x_0)\right|^p dx\right)^{1/p} \le C q_{k+1}\rho^{1+1/p},$$

$$\left(\int_I \left|\gamma_K(x) - \gamma_K(x_0)\right|^p dx\right)^{1/p} \le C q_{K+1}\rho^{1+1/p}\log(q_{K+1}),$$

$$\forall k > K,\quad \left(\int_I \gamma_k(x)^p dx\right)^{1/p} \le C\frac{\rho^{1/p}}{F_{k-K}}\left(\frac{\log(1/\rho)}{q_{K+1}} + \rho^{1/2}\right).$$

The following extension of Proposition 2 follows: If x_0 is a Diophantine number and $\varepsilon > 0$, there exists $C = C(x_0) > 0$ and $\rho_0 = \rho_0(x_0, \varepsilon) > 0$ such that for $p \ge 1$, $0 < \rho < \rho_0$,

$$\left(\frac{1}{\rho}\int_{x_0-\rho/2}^{x_0+\rho/2} |B(x_0) - B(x)|^p dx\right)^{1/p} \le C\rho^{1/(\tau(x_0)+\varepsilon)}\log(1/\rho). \tag{50}$$

From (50) we infer the lower-bound $h_B^p(x_0) \ge 1/\tau(x_0)$. Combined with (45) and Theorem 1, this yields $h_B^p(x_0) = 1/\tau(x_0)$.

5 Concluding remarks

The present paper raises the problem of determining if Theorem 1 also applies for variants of the Brjuno function.

First, B is one example of a family B_α introduced by Yoccoz [33], and further studied in [21,22]: In the definition of B, the usual continued fraction algorithm is replaced by α-continued fractions expansions, see [26]. A similar analysis as the one that we performed could be developed for B_α. Note that uniform regularity results for differences of such functions have immediate consequences on their pointwise regularity; for example, $B_{1/2} - B \in C^{1/2}$, cf. Theorem 4.6 of [22]; since the p-exponents of B belong to $[0, 1/2]$, it follows that $B_{1/2}$ shares the same p-exponent as B (except perhaps for badly approximable points where Proposition 3 leaves room for a cancellation between moduli of continuity).

Other extensions are proposed in [21] where the logarithm in the definition of B is replaced by another function. An important case consists of choosing $1/x^\beta$ with $0 < \beta < 1$; then the corresponding extension does not belong to all L^p spaces and its pointwise exponent can be studied for a restricted range of ps only. Actually, it can be seen as a fractional derivative of the corresponding Brjuno function (defined with a logarithm); we can therefore expect that (when defined) its p-exponent is $\frac{1}{\tau(x_0)} - \beta$; indeed, this would be true under the assumption that these extensions only display *cusp singularities* (i.e. if the pointwise regularity exponents of these functions are shifted by β only after a fractional integration of order β, see [2]), a plausible assumption in view of Proposition 4 which asserts that it is the case for B itself.

The Brjuno function can be interpreted as the imaginary part of a complex analytic function $\mathcal{B}$, see Section 1.3 of [23]; a remarkable property of the real part of $\mathcal{B}$ is that it is a bounded function which is continuous except at rationals, where it has a left and a right limit. This property is shared with some *Davenport series*, which are of the form $\sum a_n \omega(nx)$, where $\omega(x) = \{x\} - 1/2$ if $x \in \mathbb{R} \backslash \mathbb{Z}$ and $\omega(x) = 0$ else. If $(a_n) \in l^1$, these series display jumps located at rational numbers, thus often leading to a pointwise regularity exponent related with Diophantine approximation, see [18] in which a multifractal analysis based on the Hölder exponent is developed, and where discontinuities at rationals play a key role. This indicates that a multifractal analysis may also be performed on $Re(\mathcal{B})$: Since, for $p \in (1, \infty)$, the Hilbert transform does not modify the value of the p-exponents, see [19], it follows that (44) (and hence Theorem 1) also holds for $Re(\mathcal{B})$; thus all p-exponents of $Re(\mathcal{B})$ coincide for $p > 1$, except perhaps for $p = +\infty$. A natural conjecture therefore is that it is also the case for $p = +\infty$, i.e. that the Hölder exponent of $Re(\mathcal{B})$ is

$$\begin{cases} h_{Re(\mathcal{B})}(x_0) = 0 & \text{for } x_0 \in \mathbb{Q}, \\ h_{Re(\mathcal{B})}(x_0) = \dfrac{1}{\tau(x_0)} & \text{otherwise.} \end{cases}$$

The result clearly holds for $x_0 \in \mathbb{Q}$, because $Re(\mathcal{B})$ is discontinuous at rational points. Additionally, since any function satisfies $h_f(x_0) \le h_f^p(x_0)$, it follows that if $x_0 \notin \mathbb{Q}$, then $h_{Re(\mathcal{B})}(x_0) \le 1/\tau(x_0)$.

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