Dynamic Random Fields and Image Time Series

Analysis and Simulation from Fractional Random Field Models
Analysis from Multiplicative Interaction Models

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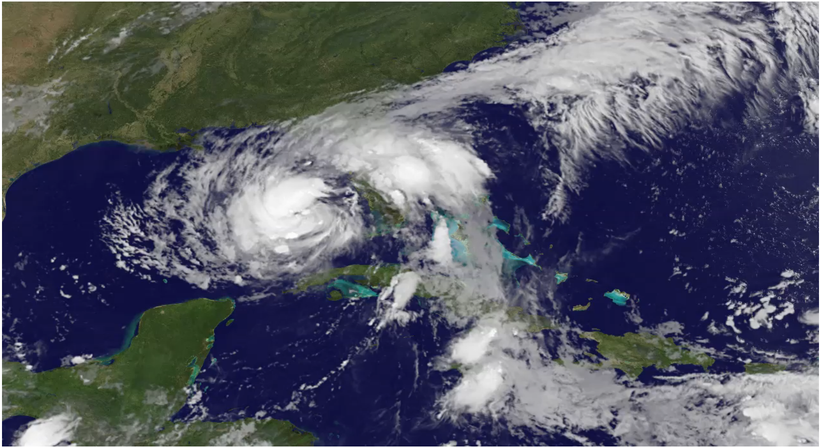
SIERRA (Signal et Images en Région Rhône-Alpes),
Annecy, FRANCE, November 6, 2014

PART I. Fractional Random Field Time Series

Introduction

Capturing, with few variables, and simulating a scene evolution in an image time series

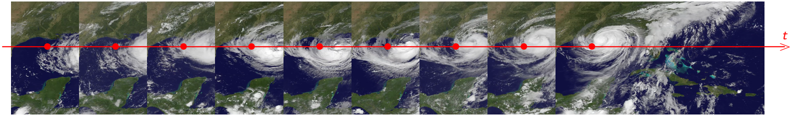


PART I. Dynamic Fractional Random Fields

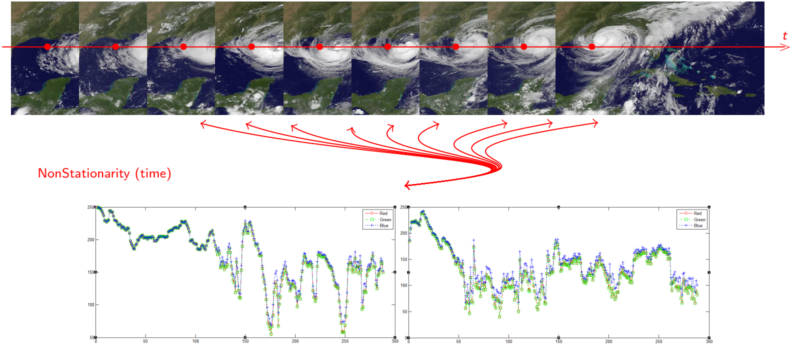
- NonStationarity
- Wavelets and (Non)Stationarity
- Random Field Time Series
- Videos

PART I. Dynamic Fractional Random Fields

NonStationarity / Image Time Series / Spatio-Temporal Dependencies



NonStationarity / Image Time Series / Spatio-Temporal Dependencies



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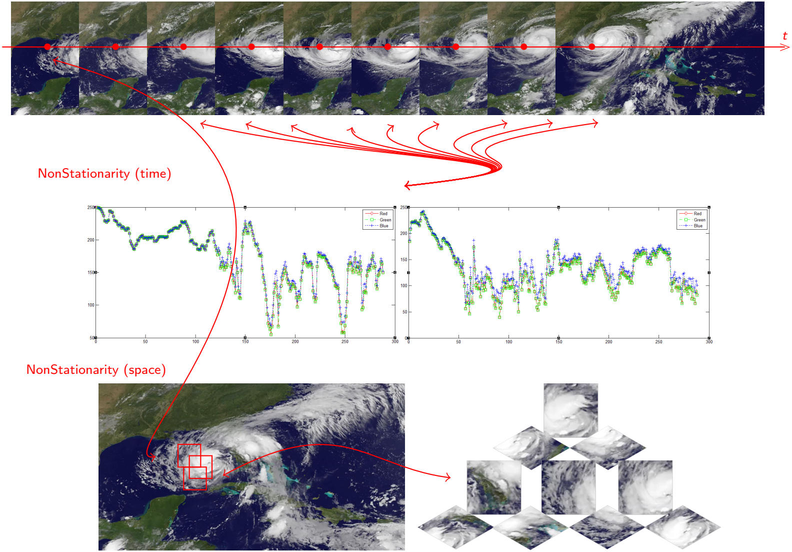

Random process / Nonstationarity / Autocorrelation Expansion

Condition on the AutoCorrelation Function, ACF

The ACF of random process X, $R(t,s) = \mathbb{E}[X(t)X(s)]$ has the following expansion:

$$R(t,s) = \underbrace{F(t) + F(s)}_{\textbf{Projective terms}} + \underbrace{\mathcal{S}(t-s)}_{\textbf{Stationary term}} + \underbrace{\sum_{1 \leqslant p,q \leqslant M} \alpha_{p,q} t^p s^q}_{\textbf{Bivariate-polynomial}} \quad (1)$$

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Example (**WSS random processes** $\leadsto$ No bivariate polynomial term)

For a **Wide Sense Stationary (WSS) random process** $X(t)$, we have

$$R_X(t,s) = R_X(t-s,0) \equiv R_X(t-s) = \mathcal{S}(t-s)$$

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Example (Polynomial random modulation $\rightsquigarrow$ No stationary term)

$(X_k)_{k=0,1,\ldots,M\#}$ are zero-mean uncorrelated random variables and

$$X(t) = \sum_{k=0}^{M\#} X_k t^k, \qquad R_X(t,s) = \sum_{k=0}^{M\#} \sigma_k^2 t^k s^k, \qquad F(t) = \sigma_0^2/2, \qquad \text{and} \qquad \mathcal{S} = 0.$$

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Example (fBm $\rightsquigarrow$ No bivariate autocorrelation polynomials)

For a **fractional Brownian motion** with Hurst parameter H, $F(t) = \mathcal{S}(t) = |t|^{2H}$.

Random Fields

Multivariate extensions of random processes

Separability and image transforms $\mathbf{Z}_{k,\ell} \equiv \mathbf{X}_k \times \mathbf{Y}_\ell$

Non-Separability characterizes a large class of natural images. We will consider non-separable extensions.

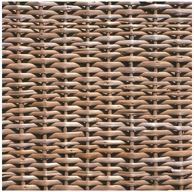
Isotropy and image transforms $\mathbf{Z}_{k,\ell} \equiv \mathbf{X}_{\sqrt{k^2+\ell^2}}$

Non-isotropy (anisotropy) dominates istotropy when considering natural images. Anisotropic field constructions will be obtained by successive directional convolutions.

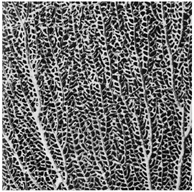
Geometry and image transforms

Geometry will be obtained by concentrating field energy in certain spectral bands.

Quasi-Separable



Non Sepable



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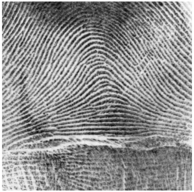
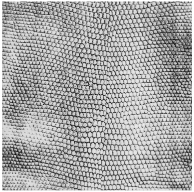
Anisotropic



Quasi-Isotropic



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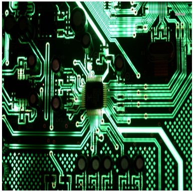
Geometry and image transforms

Geometry will be obtained by concentrating field energy in certain spectral bands.

Poor Gometry

Rich geometry



Isotropic fractional Brownian field

Let $Z_{\mathcal{H}} = Z_{\mathcal{H}}(x,y)$ be a zero-mean real valued isotropic fractional Brownian field with Hurst parameter $\mathcal{H}$, $0 < \mathcal{H} < 1$. The autocorrelation function of $Z_{\mathcal{H}}$ is $R_{Z_{\mathcal{H}}}(x,y,u,v) = \mathbb{E}\left[Z_{\mathcal{H}}(x,y)Z_{\mathcal{H}}(u,v)\right]$ with

$$R_{Z_{\mathcal{H}}}(x,y,u,v) = \frac{\sigma^2}{2}\left((x^2+y^2)^{\mathcal{H}} + (u^2+v^2)^{\mathcal{H}}\right) - \frac{\sigma^2}{2}\left[(x-u)^2+(y-v)^2\right]^{\mathcal{H}}.$$

Stationary (S) and Projective (P) terms :

$$S(t,s) = -\frac{\sigma^2}{2}\left[t^2+s^2\right]^{\mathcal{H}} = P(t,s)$$

A. Yaglom, *Some classes of random fields in N-D space, related to stationary random processes*, Theory Proba. Appl. (1957)
B. Pesquet-Popescu, J. Lévy-Véhel, *Stochastic fractal models for image processing*, IEEE SP Magazine (2002)

NonStationarity

Modulated Isotropic fractional Brownian field

Random field $G_{\mathcal{H}(q)}$ is the modulation of an isotropic fractional Brownian random field $Z_{\mathcal{H}(q)}$ by using a complex exponential with frequency point $(u_q, v_q) \in [0, \pi] \times [0, \pi]$.

$$G_{\mathcal{H}(q)}(t, s) = e^{iu_q t} e^{iv_q s} Z_{\mathcal{H}(q)}(t, s). \quad (2)$$

- Random field $G_{\mathcal{H}(q)}$ has autocorrelation function

$$R_{G_{\mathcal{H}(q)}}(t, s, x, y) = \mathbb{E}\left[G_{\mathcal{H}(q)}(t, s)\overline{G_{\mathcal{H}(q)}(x, y)}\right] = R_{Z_{\mathcal{H}(q)}}(t, s, x, y) e^{iu_q(t-x)} e^{iv_q(s-y)}. \quad (3)$$

- Random field $G_{\mathcal{H}(q)}$ is non-stationary.

Generalized Q-factor fractional Brownian field

Consider a sequence of Hurst parameters $\mathcal{H}_Q = \{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}(q)\}$ and
Define a generalized fractional field from a convolution of Q independent and non-stationary fields $\mathcal{G} = \{G_{\mathcal{H}(q)}, \text{q} = 1, 2, \ldots, \text{Q}\}$:

$$\mathcal{E}_{\mathcal{H}_Q} = \bigotimes_{q=1}^{Q} G_{\mathcal{H}(q)} \quad (4)$$

- The Q-factor *Generalized FBF* (GFBF) $\mathcal{E}_{\mathcal{H}_Q}$ is a non-stationary random field.

A. M. Atto, Z. Tan, O. Alata, M. Moreaud, *Non-Stationary Texture Synthesis from Random Field Modeling*, IEEE ICIP (2014)

Outline

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Outline

Wavelet Packets (WP)

Separable version

Functional space $\mathbf{W}_{j,[n_1,n_2]}$ (WP subband) spanned by orthogonal *wavelet packet functions*:

$$\left\{\tau_{[2^j k_1, 2^j k_2]} W_{j,[n_1,n_2]} : (k_1, k_2) \in \mathbb{Z}^2\right\},$$

where $W_{0,0_N} = \phi$ is a *scaling function*, τ is the *shift operator* and the *wavelet packet function*, $W_{j,[n_1,n_2]}$, satisfies in the Fourier domain (notation $\mathcal{F}$):

$$\mathcal{F}W_{j,[n_1,n_2]}(\omega_1, \omega_2) = \mathcal{F}W_{j,n_1}(\omega_1)\mathcal{F}W_{j,n_2}(\omega_2), \tag{5}$$

$$\mathcal{F}W_{j,[n_1,n_2]} = \mathbf{H}_{j,[n_1,n_2]}\mathcal{F}\Phi, \tag{6}$$

the *multiscale wavelet packet filter*, $\mathbf{H}_{j,[n_1,n_2]}$, satisfies

$$\mathbf{H}_{j,[n_1,n_2]}(\omega_1, \omega_2) = \prod_{i=1}^{2} \mathbf{H}_{j,n_i}(\omega_i), \quad \mathbf{H}_{j,n_i}(\omega) = 2^{j/2}\left[\prod_{\ell=1}^{j} H_{\epsilon_\ell^i}(2^{\ell-1}\omega)\right]$$

for $\epsilon_\ell^i \in \{0, 1\}$, where H_0 and H_1 are standard *scaling filter* (H_0) and *wavelet filter* (H_1).

The subband $\mathbf{W}_{j,[n_1,n_2]}$ coefficients of Z define a random field $c_{j,[n_1,n_2]}$,

$$c_{j,[n_1,n_2]}[k_1, k_2] = \int_{\mathbb{R}^N} Z(\bullet)\tau_{2^j[k_1,k_2]} W_{j,[n_1,n_2]}(\bullet)\mathrm{d}\bullet$$

Wavelet Packets & NonStationarity

- $X_t \rightsquigarrow$ zero-mean 2^{nd} order random field, with $t \in \mathbb{R}^N$.
- $X_t \rightsquigarrow$ continuous in quadratic mean.

Condition (ACF) / on the autocorrelation function expansion

The AutoCorrelation Function, ACF, can be written in the form:

$$R(t,s) = \underbrace{F(t) + F(s)}_{\textbf{Projective terms}} + \underbrace{\mathcal{S}(t-s)}_{\textbf{Stationary term}} + \underbrace{\sum_{1\leqslant p,q \leqslant M} \alpha_{p,q} t^p s^q}_{\textbf{Bivariate-polynomial}}$$

$$\text{with} \quad F(t) W_{j,n,k}(t) \in L^1(\mathbb{R}^N), \quad \text{and} \quad t^p s^q W_{j,n,k}(t) W_{j,n,\ell}(s) \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$$

for every $1 \leqslant p,q \leqslant M$, where $\mathcal{S}$ is an even function.

Theorem (wide sense stationarity)

Let r be the wavelet order. Under Condition **(ACF)**, the discrete random sequence $c^r_{j,n}$, $n \neq 0$, is wide sense stationary for $r \geqslant M+1$: $R^r_{j,n}[k,\ell] \equiv R^r_{j,n}[k-\ell]$, with

$$R^r_{j,n}[m] = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}\mathcal{S}(\omega) \left| \mathcal{F} W^r_{j,n}(\omega) \right|^2 e^{i2^j m\omega} d\omega. \quad (7)$$

A. M. Atto, Y. Berthoumieu, *Wavelet Transforms of Nonstationary Random Processes: Contributing Factors for Stationarity and Decorrelation*, IEEE T-IT, Vol. 58, No. 1 (2012)

Wavelet Packets & NonStationarity

Statistical Properties of the Wavelet Packet Coefficients of Random Processes

Asymptotic properties of the discrete random process $c_{j,n}^{[r]}$?

Theorem (Decorrelation - Independence)

Let $\mathcal{P} = \left(h^r_{\epsilon_\ell}\right)_\ell = (\mathbf{U}^r, \{\mathbf{W}^r_{j,n_{\mathcal{P}}(j)}\}_{j\in\mathbb{N}})$ be a path in the wavelet packet tree. Assume that $\mathcal{P} \neq \mathcal{P}_0$ where $\mathcal{P}_0$ is the approximation path.

- Assume $\mathcal{FS}$ is continuous at the frequency $\omega_{\mathcal{P}}$ defined by

$$\omega_{\mathcal{P}} = \lim_{j\to+\infty} \frac{G(n_{\mathcal{P}}(j))\pi}{2^j},$$

  where G is a separable permutation recursively defined by $G(2\ell+\epsilon) = 3G(\ell) + \epsilon - 2\left\lfloor \frac{G(\ell)+\epsilon}{2} \right\rfloor$.
- Then, the autocorrelation $R^r_{j,n_{\mathcal{P}}(j)}$ of $c^r_{j,n_{\mathcal{P}}(j)}$ uniformly satisfies:

$$\lim_{j\to+\infty}\left(\lim_{r\to+\infty} R^r_{j,n_{\mathcal{P}}(j)}[k]\right) = \mathcal{FS}(\omega_{\mathcal{P}})\delta[k]. \quad (8)$$

A. M. Atto, Y. Berthoumieu, *Wavelet Transforms of Nonstationary Random Processes: Contributing Factors for Stationarity and Decorrelation*, IEEE T-IT, Vol. 58, No. 1 (2012)

Wavelet Packets and the Fractional Brownian Motion

Fractional Brownian Motion: only one vanishing moment is required for stationarity of its the wavelet coefficients (the scaling coefficients remains non-stationary).

Example of fractional Brownian motions

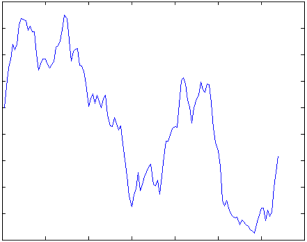
- **fBm random process**: If X is a fractional Brownian motion with Hurst parameter H, $0 < H < 1$, then

$$\mathcal{FS}(\omega) = \frac{\sigma^2 D(H)}{|\omega|^{2H+1}}, \qquad (9)$$

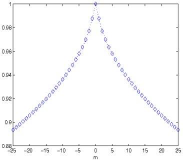
  where $D(H) = \Gamma(2H+1)\sin(\pi H)$ and Γ is the standard Gamma function.

- **Spectrum** $\mathcal{FS}$ has 1 singularity point.

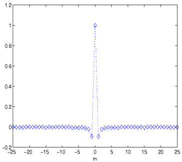
fBm, $H = 0.75$, Spectrum $C/|\omega|^{2.5}$



$R^{[7]}_{6,0}[m]$



$R^{[1]}_{6,2^6-3}[m]$



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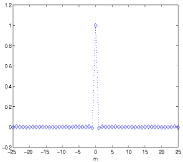


Wavelet Packets and the (Generalized) Fractional Brownian Fields

- WP spectrum of GFBFs: spectral poles at different frequency location ($\rightsquigarrow$ paths).

Fractional Brownian Field — 1 spectral pole located at 0.

$$\gamma_{Z_{\mathcal{H}}}(u,v) = \xi(\mathcal{H}) \frac{1}{(u^2+v^2)^{\mathcal{H}+1}}, \qquad // \quad \xi(z) = \frac{2^{-(2z+1)}\pi^2\sigma^2}{\sin(\pi z)\Gamma^2(1+z)}. \tag{10}$$

Fractional Brownian Field Modulation — 1 spectral pole located at (u_q, v_q).

$$\gamma_{G_{\mathcal{H}(q)}}(u,v) = \xi(\mathcal{H}(q)) \frac{1}{\left((u-u_q)^2+(v-v_q)^2\right)^{\mathcal{H}(q)+1}}. \tag{11}$$

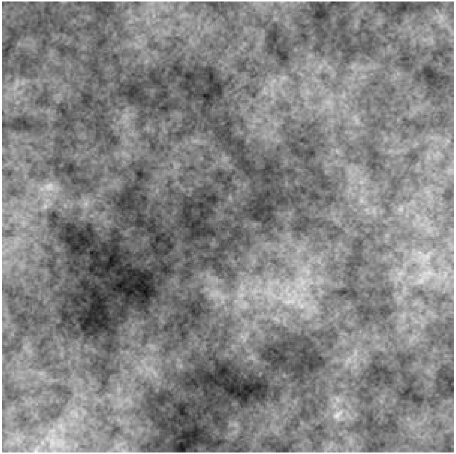
Generalized Fractional Brownian Field — Several spectral poles.

$$\gamma_{\mathcal{E}_{\mathcal{H}_Q}}(u,v) \propto \prod_{q=1}^{Q} \frac{\xi(\mathcal{H}(q))}{\left[(u-u_q)^2+(v-v_q)^2\right]^{\mathcal{H}(q)+1}}. \tag{12}$$

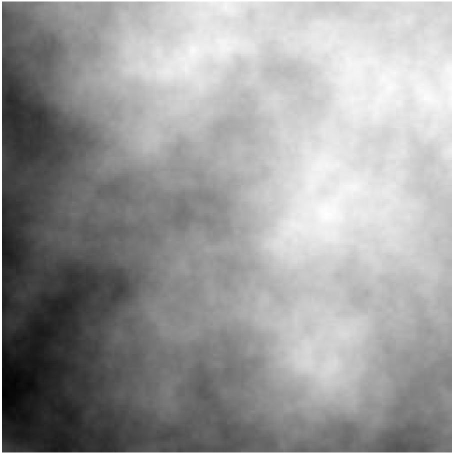
A. M. Atto, Z. Tan, O. Alata, M. Moreaud, *Non-Stationary Texture Synthesis from Random Field Modeling*, IEEE ICIP (2014)

Examples of FBF (one spectral pole located at 0).

Z, $\mathcal{H} = 0.2$

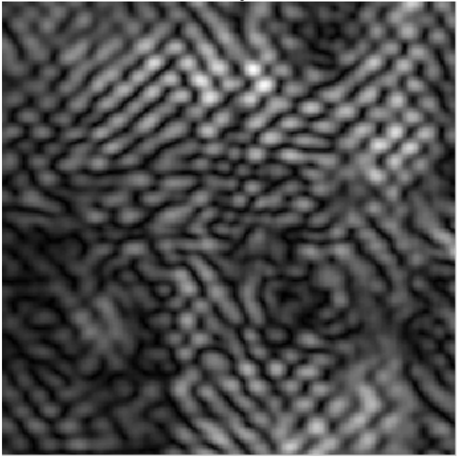


Z, $\mathcal{H} = 0.75$

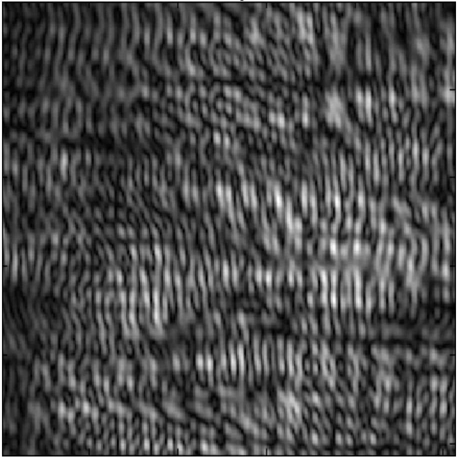


Examples of GFBFs with 3 and 6 spectral poles respectively.

$\mathcal{E}_{\mathcal{H}_3}$

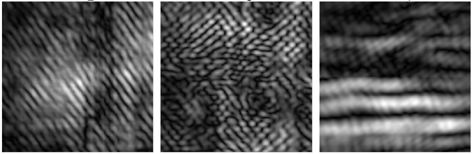


$\mathcal{E}_{\mathcal{H}_6}$



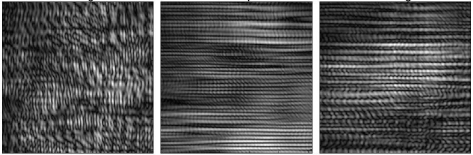
Gaussian poles and uniform Hurst parameters

$\mathcal{E}_{\mathcal{H}_2}$ $\mathcal{E}_{\mathcal{H}_3}$ $\mathcal{E}_{\mathcal{H}_4}$



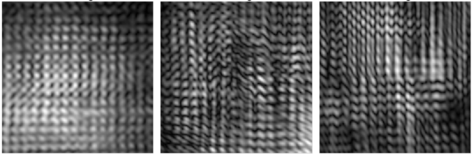
Uniform poles and uniform Hurst parameters

$\mathcal{E}_{\mathcal{H}_6}$ $\mathcal{E}_{\mathcal{H}_7}$ $\mathcal{E}_{\mathcal{H}_8}$



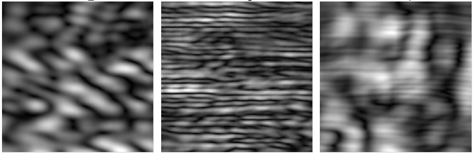
Uniform poles and constant Hurst sequence

$\mathcal{E}_{\mathcal{H}_6}$ $\mathcal{E}_{\mathcal{H}_6}$ $\mathcal{E}_{\mathcal{H}_6}$

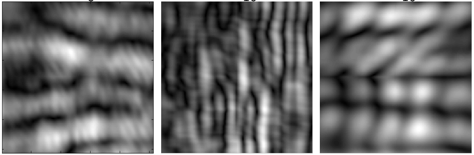


Gamma poles and uniform Hurst parameters

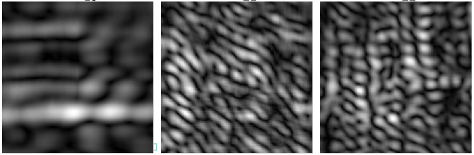
$\mathcal{E}_{\mathcal{H}_2}$ $\mathcal{E}_{\mathcal{H}_3}$ $\mathcal{E}_{\mathcal{H}_4}$



$\mathcal{E}_{\mathcal{H}_8}$ $\mathcal{E}_{\mathcal{H}_{10}}$ $\mathcal{E}_{\mathcal{H}_{16}}$



$\mathcal{E}_{\mathcal{H}_{19}}$ $\mathcal{E}_{\mathcal{H}_{21}}$ $\mathcal{E}_{\mathcal{H}_{22}}$



Wavelet Packet "Stationarization" $\rightsquigarrow$ Power Spectral Density estimation

2D Wavelet Packet Spectrum

- Set: $P = \left\{0, 1, \ldots, 2^j - 1\right\} \times \left\{0, 1, \ldots, 2^j - 1\right\}$ of frequency indices.
- Apply G^{-1} to P to obtain a new (reordered) grid N composed with indices $(n_1, n_2) = \left(G^{-1}(p_1), G^{-1}(p_2)\right)$, where

  the permutation G is defined by $G(0) = 0$ and $G(2\ell + \epsilon) = 3G(\ell) + \epsilon - 2\left\lfloor \frac{G(\ell) + \epsilon}{2} \right\rfloor$. (13)

- Compute: binary sequences $(\epsilon^1_\ell)_{\ell=1,2,\ldots,j}, (\epsilon^2_\ell)_{\ell=1,2,\ldots,j} \in \{0, 1\}^j$ associated with elements (n_1, n_2) of N, from

$$\left(n_1 = \sum_{\ell=1}^{j} \epsilon^1_\ell 2^{j-\ell} \qquad // \qquad n_2 = \sum_{\ell=1}^{j} \epsilon^2_\ell 2^{j-\ell} \right). \tag{14}$$

- Compute: quaternary sequences $(\mu_\ell)_{\ell=1,2,\ldots,j} \in \{0, 1, 2, 3\}^j$ associated with elements of grid N, from

$$\mu = 2\epsilon^1 + \epsilon^2 = \begin{cases} 0 & \text{if} \quad (\epsilon^1, \epsilon^2) = (0, 0), \\ 1 & \text{if} \quad (\epsilon^1, \epsilon^2) = (0, 1), \\ 2 & \text{if} \quad (\epsilon^1, \epsilon^2) = (1, 0), \\ 3 & \text{if} \quad (\epsilon^1, \epsilon^2) = (1, 1). \end{cases} \tag{15}$$

- Compute: the set of frequency indices $n \in \left\{0, 1, \ldots, 4^j - 1\right\}$ associated with grid N, according to

$$n = \sum_{\ell=1}^{j} \mu_\ell 4^{j-\ell}. \tag{16}$$

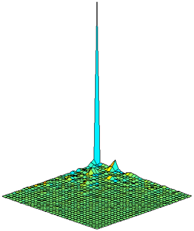
- Replace, in grid N, every pair (n_1, n_2) by its corresponding n obtained from steps above.
- Set: for every $(p_1, p_2) \in P$ and the corresponding $n \in N$,

$$\widehat{\gamma}\left(\frac{p_1 \pi}{2^j}, \frac{p_2 \pi}{2^j}\right) = \mathrm{Var}[c_{j,n}]. \tag{17}$$

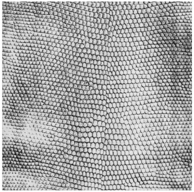
A. M. Atto, Y. Berthoumieu, P. Bolon, *2-D Wavelet Packet Spectrum for Texture Analysis*, IEEE T-IP, vol. 22, no. 6 (2013)

Wavelet Packet "Stationarization" $\rightsquigarrow$ Power Spectral Density estimation

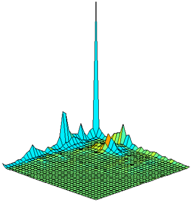
FT-PSD "D3"



"D3" / Brodatz



WP-PSD "D3"



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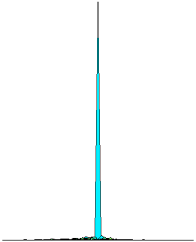
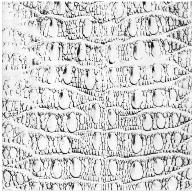
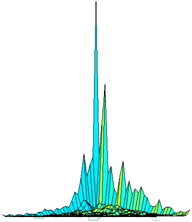


"D10" / Brodatz

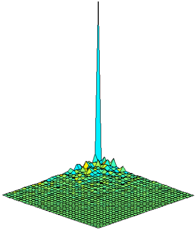


WP-PSD "D10"

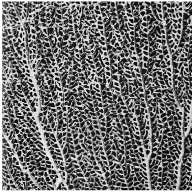


Wavelet Packet "Stationarization" $\rightsquigarrow$ Power Spectral Density estimation

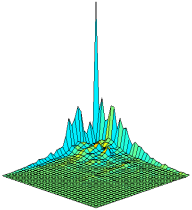
FT-PSD "D87"



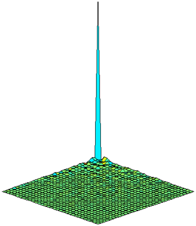
"D87" / Brodatz



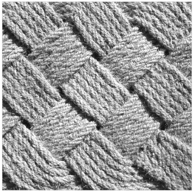
WP-PSD "D87"



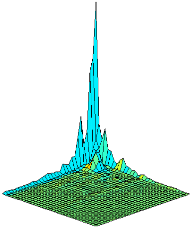
FT-PSD "Fabric.11"



"Fabric.11" / VisTeX

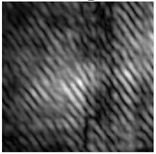


WP-PSD "Fabric.11"

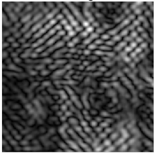


Gaussian poles and uniform Hurst parameters

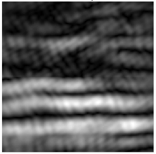
$\mathcal{E}_{\mathcal{H}_2}$



$\mathcal{E}_{\mathcal{H}_3}$

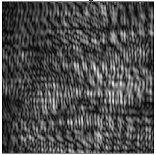


$\mathcal{E}_{\mathcal{H}_4}$

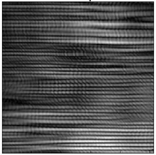


Uniform poles and uniform Hurst parameters

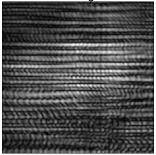
$\mathcal{E}_{\mathcal{H}_6}$



$\mathcal{E}_{\mathcal{H}_7}$

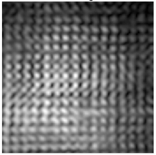


$\mathcal{E}_{\mathcal{H}_8}$

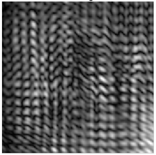


Uniform poles and constant Hurst sequence

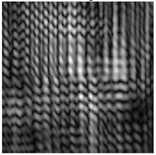
$\mathcal{E}_{\mathcal{H}_6}$



$\mathcal{E}_{\mathcal{H}_6}$

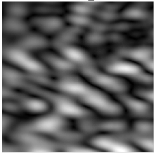


$\mathcal{E}_{\mathcal{H}_6}$

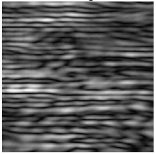


Gamma poles and uniform Hurst parameters

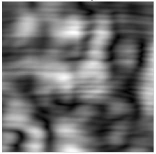
$\mathcal{E}_{\mathcal{H}_2}$



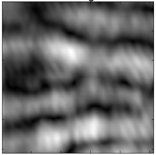
$\mathcal{E}_{\mathcal{H}_3}$



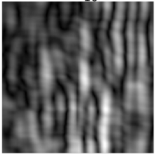
$\mathcal{E}_{\mathcal{H}_4}$



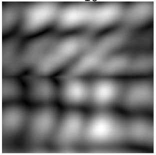
$\mathcal{E}_{\mathcal{H}_8}$



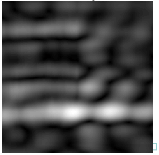
$\mathcal{E}_{\mathcal{H}_{10}}$



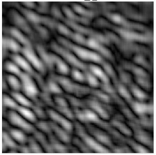
$\mathcal{E}_{\mathcal{H}_{16}}$



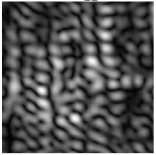
$\mathcal{E}_{\mathcal{H}_{19}}$



$\mathcal{E}_{\mathcal{H}_{21}}$

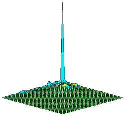


$\mathcal{E}_{\mathcal{H}_{22}}$



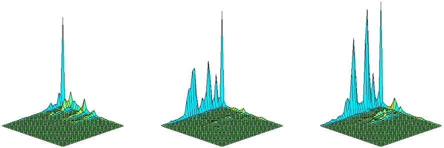
Gaussian poles and uniform Hurst parameters

$\mathcal{S}_W\mathcal{E}_{\mathcal{H}_2}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_3}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_4}$



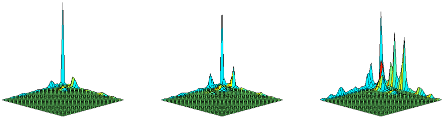
Uniform poles and uniform Hurst parameters

$\mathcal{S}_W\mathcal{E}_{\mathcal{H}_6}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_7}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_8}$



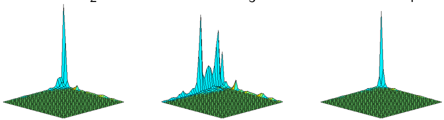
Uniform poles and constant of Hurst sequence

$\mathcal{S}_W\mathcal{E}_{\mathcal{H}_6}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_6}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_6}$

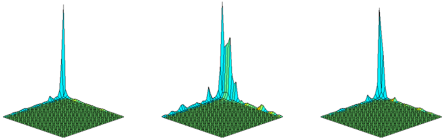


Gamma poles and uniform Hurst parameters

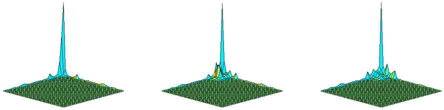
$\mathcal{S}_W\mathcal{E}_{\mathcal{H}_2}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_3}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_4}$



$\mathcal{S}_W\mathcal{E}_{\mathcal{H}_8}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_{10}}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_{16}}$



$\mathcal{S}_W\mathcal{E}_{\mathcal{H}_{19}}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_{21}}$ $\mathcal{S}_W\mathcal{E}_{\mathcal{H}_{22}}$



Outline

- NonStationarity
- Wavelets and (Non)Stationarity
- Random Field Time Series
- Videos

Outline

Wavelet Packet spatio-temporal spectrum warping

Simulation of Fractional Field Time Series

Let Z_{α_0} be an fBf with WP coefficients $(c_{j,[n_1,n_2]})_{j,[n_1,n_2]}$ and associated spectrum γ_0. Let $0 < \alpha(t) < 1$. Define

$$\Theta_{j,[n_1,n_2]}(t) = \frac{\xi^{\frac{1}{2}}(\alpha(t))}{\xi^{\frac{1}{2}\frac{\alpha(t)+1}{\alpha_0+1}}(\alpha_0)} \gamma_0^{\frac{1}{2}\frac{\alpha(t)-\alpha_0}{\alpha_0+1}} \left(\frac{p_1\pi}{2^j}, \frac{p_2\pi}{2^j}\right), \tag{18}$$

$$d_{j,[n_1,n_2]}[k_1,k_2] = \Theta_{j,[n_1,n_2]}(t) c_{j,[n_1,n_2]}[k_1,k_2], \tag{19}$$

$$\Upsilon_{j,[n_1,n_2]}(x,y) = \sum_{k_1,k_2\in\mathbb{Z}} d_{j,[n_1,n_2]}[k_1,k_2] \tau_{2^j[k_1,k_2]} W^{\boxed{-1}}_{j,[n_1,n_2]}(x,y). \tag{20}$$

$$\Upsilon_t(x,y) = \sum_{n_1,n_2=0}^{2^j-1} \Upsilon_{j,[n_1,n_2]}(x,y) \tag{21}$$

is a *fractional Brownian field with Hurst parameter* $\alpha(t)$ and associated WP spectrum

$$\gamma_t = \frac{\xi(\alpha(t))}{\xi^{\frac{\alpha(t)+1}{\alpha_0+1}}(\alpha_0)} \gamma_0^{\frac{\alpha(t)+1}{\alpha_0+1}} \tag{22}$$

■ A. M. Atto and L. Fillatre and M. Antonini and I. Nikiforov, *Simulation of Image Time Series from Dynamical Fractional Brownian Fields*, IEEE ICIP (2014)

Image Time Series Synthesis from WP spectrum warping

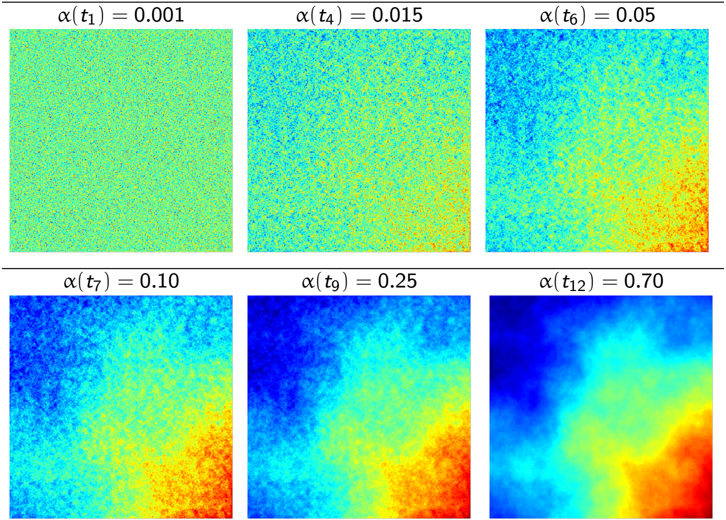


Image Time Series Synthesis from WP spectrum warping

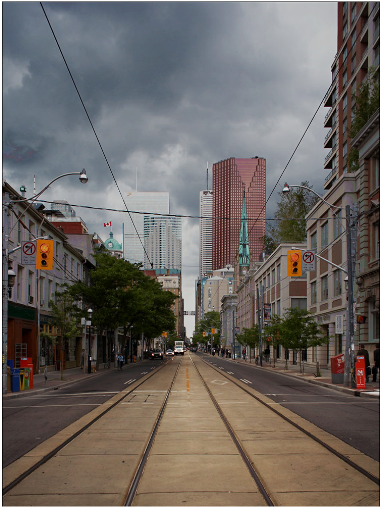
$\alpha(t_1) = 0.45$	$\alpha(t_2) = 0.50$
	

Image Time Series Synthesis from WP spectrum warping

$\alpha(t_2) = 0.50$

$\alpha(t_3) = 0.55$

Image Time Series Synthesis from WP spectrum warping

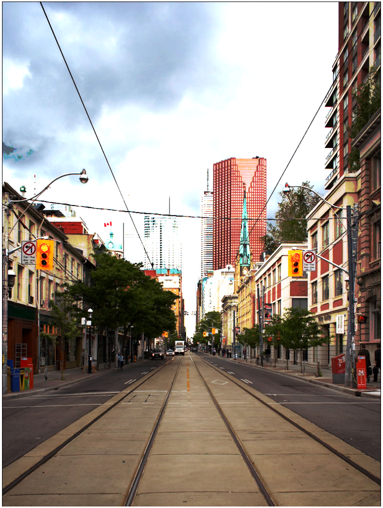
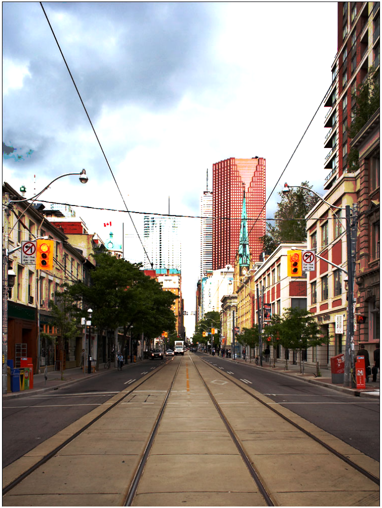
$\alpha(t_3) = 0.55$	$\alpha(t_4) = 0.60$
	

Image Time Series Synthesis from WP spectrum warping

$\alpha(t_4) = 0.60$



$\alpha(t_1) = 0.45$

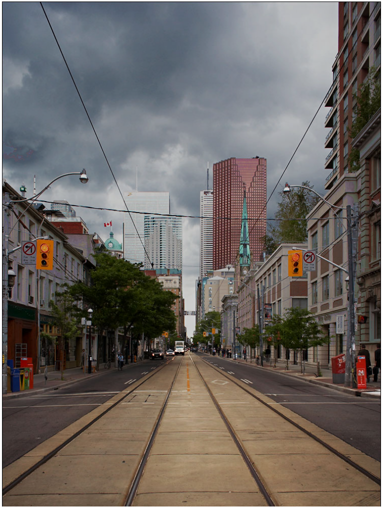
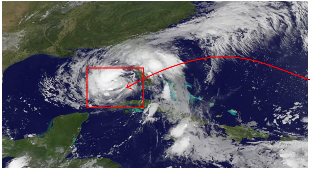
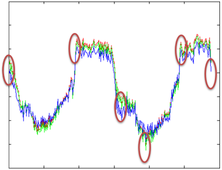


Image Time Series Analysis from WP spectrum power decay

ROI ISAAC Hurricane

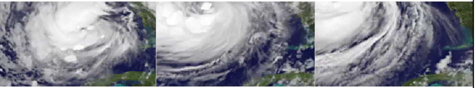


Time Series of Spatial “Hurst” Parameters



Critical dates for Hurst parameter time series

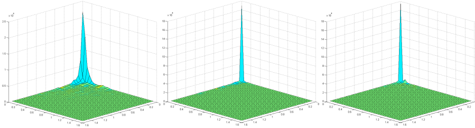
Date 1 95 153



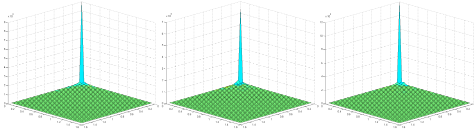
195 243 288

Spectral information of critical dates

1 95 153



195 243 288



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Outline

- NonStationarity
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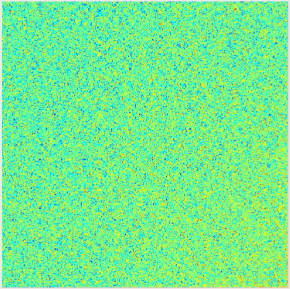
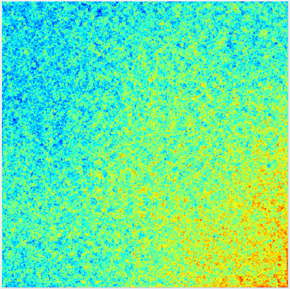
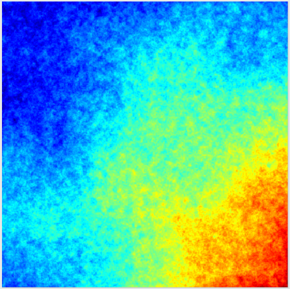
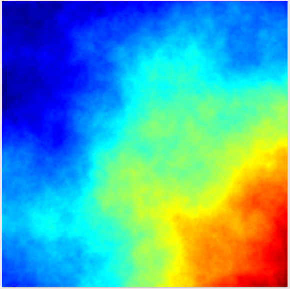
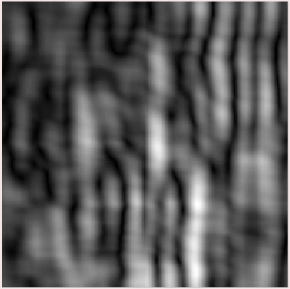
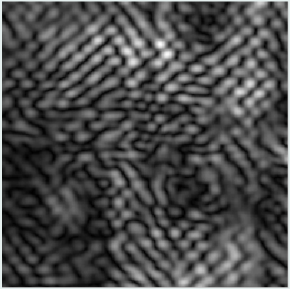


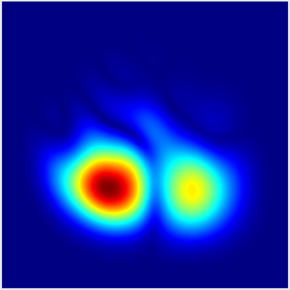


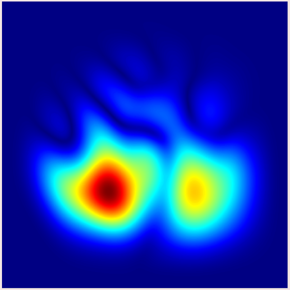


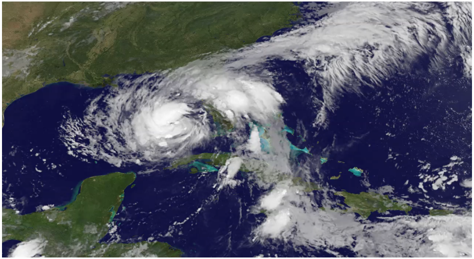


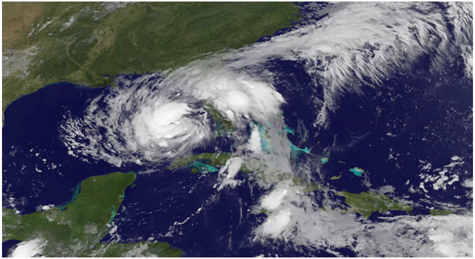












PART II. Multiplicative Interacting Random Field Time Series

A. M. Atto, E. Trouvé, J.-M. Nicolas, *Sparsity Information and Multiplicative Observation Models*, Preprint (2014)

PART II. (SAR) Random Fields in Multiplicative Environment

- Sampling in time (increasingly close acquisition dates)

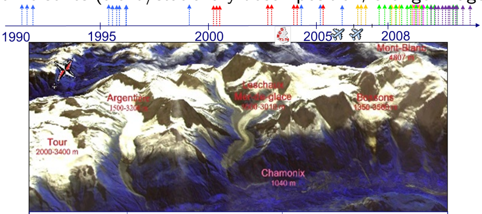
[TerraSAR-X (TS-X)]: 11 day acquisition cycle.

- Spatial resolution (different acquisition modalities and spectral bands).

Satellite – *[TS-X]*: resolution $\cong$ 1m , scene with size 5-10km $\times$ 5km.
Airborne – *[FSAR]*: resolution $\cong$ 0.25 m, pixel spacing 15 cm $\times$ 17cm.

- Stochasticity in multiplicative algebra (speckle effect).

Random field image time series (trend/stationary decomposition, change-image, *etc.*).



	Argentière	Mer de Glace – Leschaux
Localisation	45°56'15'"N / 7°00'30" E	45°55'15" N/ 6°55'45" E
Surface / Length	15 (km²) / 9 (km)	3,5 (km²) / 4.7 (km)
Mean slope	~14° (26%)	~ 9° (17%)

PART II. (SAR) Random Fields in Multiplicative Environment

PART II. (SAR) Random Fields in Multiplicative Environment

- SAR Model & Stationarity
- Geometric Wavelets and SAR
- SAR ITS Analysis
- Conclusion

SAR Model & Stationarity

Multiplicative interaction model in a sparsifying domain context

Function f is sparse in $\mathcal{W}$ domain, where $\mathcal{W}$ is a linear transform, but we observe:

$$f(k)X(k) = \begin{cases} f(k) + f(k)[X(k) - 1] & (\star) \\ e^{\log f(k) + \log X(k)} & (\star\star) \end{cases}$$

where

- f and X are strictly positive function and random sequence,
- $(X(k))_k$ are unitary-mean, stationary random variables,
- X is independent with f.

Which model?

Given f and $\mathcal{W}$, who is the lesser between the two blue evils present in $(\star)$, $(\star\star)$ for the $\mathcal{W}$-domain representation of $f(k)X(k)$?

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SAR Model, Wavelets & Stationarity

Model ($\star$): additive noise model with signal-dependent noise

The additive 'noise' in model ($\star$) is a random sequence Z:

$$Y(k) = f(k)[X(k) - 1].$$

Let $\mathcal{W} = \mathbf{W}_{j,n}$ be a wavelet subband. We have: $\mathcal{W}Y$ is non-stationary in general, excepted some few cases, for instance when

- f is constant, or
- f is polynomial with order r and the $\mathcal{W}$ generating functions have at least r vanishing moments.

Model ($\star\star$): 'additive' noise in a multiplicative algebra

Wavelet in a multiplicative algebra [binary operations $(\oplus, \otimes) = (\times, \wedge)$] or *geometric wavelet*, $\mathcal{W}^G$, involves geometric approximations and differencing, with:

$$\mathcal{W}^G fX = [\mathcal{W}^G f] \oplus [\mathcal{W}^G X].$$

and $\mathcal{W}^G X$ inherits the *stationarity* property of X.

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Geometric wavelets / Geometric convolution

Multiplicative algebra

Consider a data sequence $\mathbf{x} = (\mathbf{x}[\ell])_{\ell \in \mathbb{Z}}$, with $\mathbf{x}[\ell] \in \mathbb{R}^+$ for every $\ell \in \mathbb{Z}$. Assume that this sequence represents a multiplicative 'process'. Then:

- "zero" or "nothing" or "no change" corresponds to identity element "1"
- a "small" value is a value close to 1 (10^{-3} and 10^3 have the same significance in terms of *absolute proportion*,
- The support of $\mathbf{x}$ is $\{\ell \in \mathbb{Z} : \mathbf{x}[\ell] \neq 1\}$,
- Consequence: a missing value must be replaced by 1.

Geometric convolution

Let $\mathbf{h} = (\mathbf{h}[\ell])_{\ell \in \mathbb{Z}}$ denotes the impulse response of a digital filter. We define the geometric convolution of $\mathbf{x}$ and $\mathbf{h}$ on the vectorial space $(\mathbb{R}^+, \times, \wedge)$ as:

$$\begin{aligned} \mathbf{y}[k] = \mathbf{x} \circledast \mathbf{h}[k] &\triangleq \prod_{\ell \in \mathbb{Z}} (\mathbf{x}[\ell])^{\mathbf{h}[k-\ell]} \\ &= \prod_{\ell \in \mathbb{Z}} (\mathbf{x}[k-\ell])^{\mathbf{h}[\ell]} \triangleq \mathbf{h} \circledast \mathbf{x}[k], \end{aligned}$$

Geometric wavelets / Geometric convolution

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Geometric wavelets / Definition

Geometric wavelet decomposition

Define the *geometric wavelet decomposition* of $\mathbf{x}$ by:

$$\mathbf{c}_{1,0}[k] = \mathbf{x} \circledast \overline{\mathbf{h}_0}[2k], \tag{23}$$

$$\mathbf{c}_{1,1}[k] = \mathbf{x} \circledast \overline{\mathbf{h}_1}[2k], \tag{24}$$

and, recursively, for $\epsilon \in \{0,1\}$ (geometric approximations when $\epsilon = 0$ and geometric differences/details when $\epsilon = 1$):

$$\mathbf{c}_{j+1,2n+\epsilon}[k] = \mathbf{c}_{j,n} \circledast \overline{\mathbf{h}_\epsilon}[2k]. \tag{25}$$

Geometric wavelet reconstruction

We have:

$$\mathbf{c}_{j,n}[k] = \left(\check{\mathbf{c}}_{j+1,2n} \circledast \mathbf{h}_0[k]\right) \times \left(\check{\mathbf{c}}_{j+1,2n+1} \circledast \mathbf{h}_1[k]\right). \tag{26}$$

where

$$\check{\mathbf{u}}[2k+\epsilon] = \begin{cases} \mathbf{u}[k] & \text{if} \quad \epsilon = 0, \\ 1 & \text{if} \quad \epsilon = 1. \end{cases} \tag{27}$$

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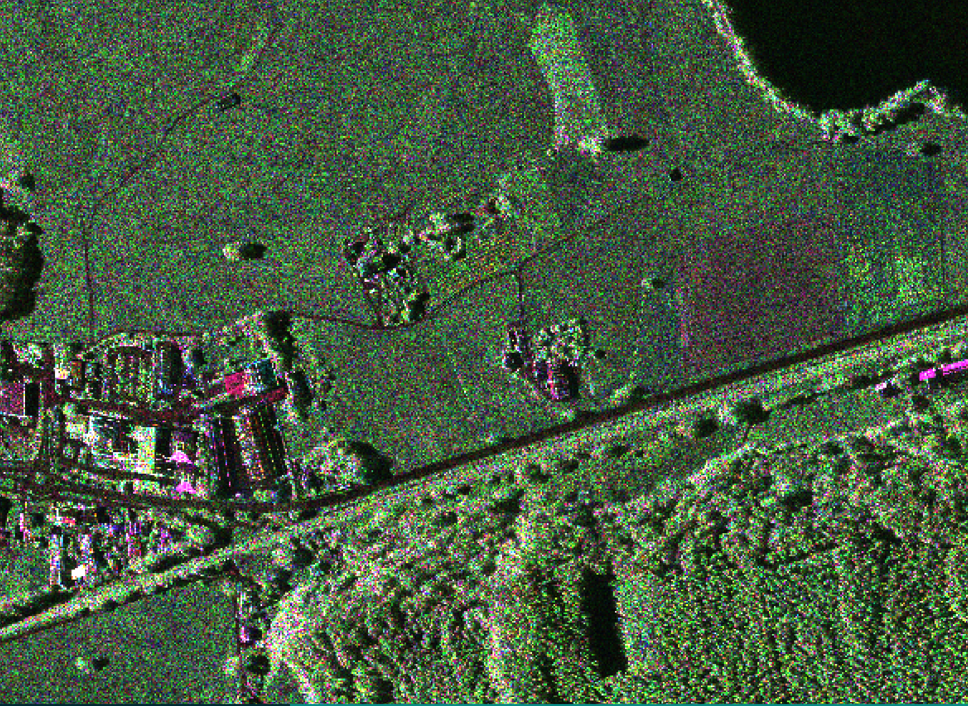
Geometric wavelets / Implementation

Geometric wavelet decomposition implementation

- Define a variable environment and a data type where
  - '0' (of the standard type) behaves as '1'
  - '$+$ calls' involve $\times$ operation,
  - '$\times$ calls' involve $\wedge$ operation.
- Call this environment and use your 'standard' additive tools

OR

- compute the *log* function of the input data,
- apply 'standard' additive wavelets to the *log* of data,
- apply *exp* function to wavelet coefficients.



Which model (for sparsity)?

"+" wavelets - DiagDet J=1

"×" wavelets - DiagDet J=1

Which model (for sparsity)?

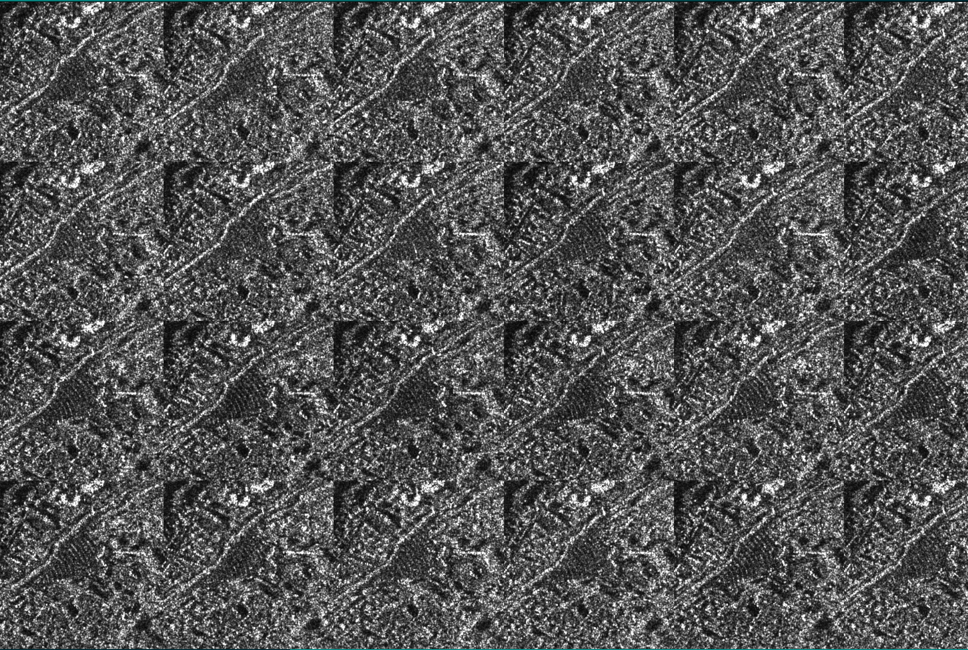
"+" wavelets - DiagDet J=3

"$\times$" wavelets - DiagDet J=3

Which model (for sparsity)?

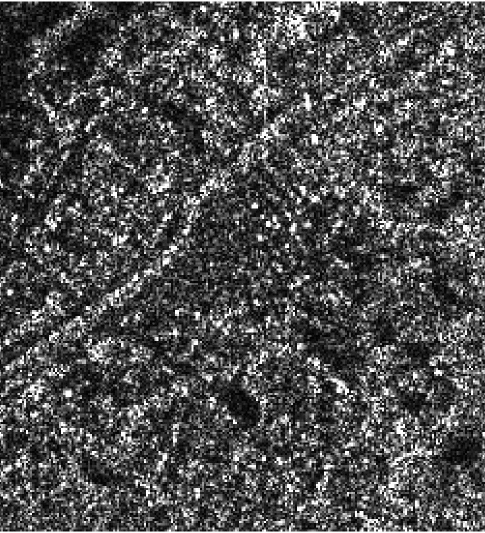
"+" wavelets - Approx J=3

"×" wavelets - Approx J=3

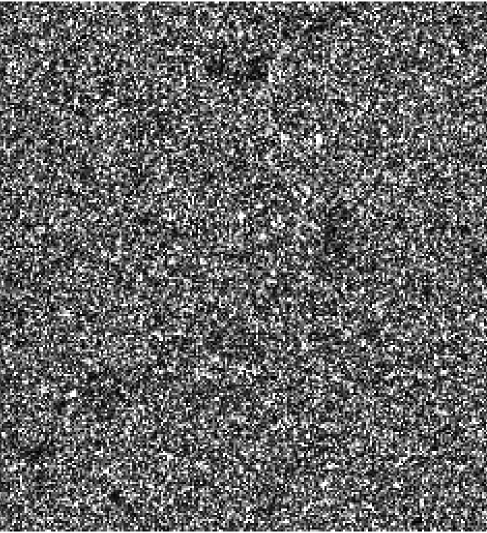


Which model (for sparsity)?

“+” wavelets - TempDet - db12

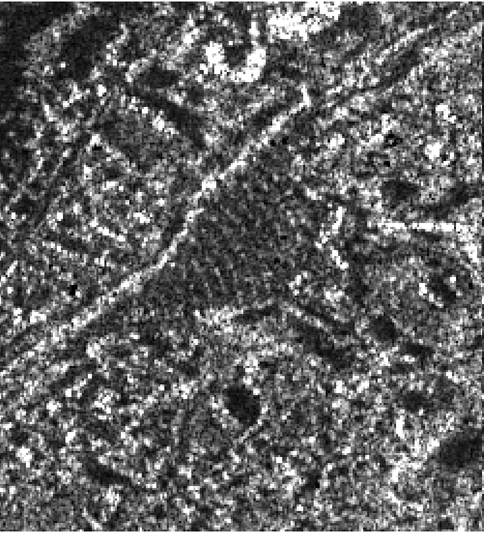


“×” wavelets - TempDet - db12

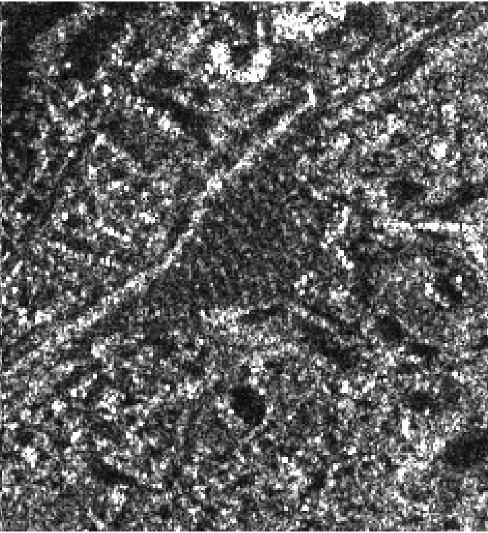


Which model (for sparsity)?

"+" wavelets - TempApp - db12

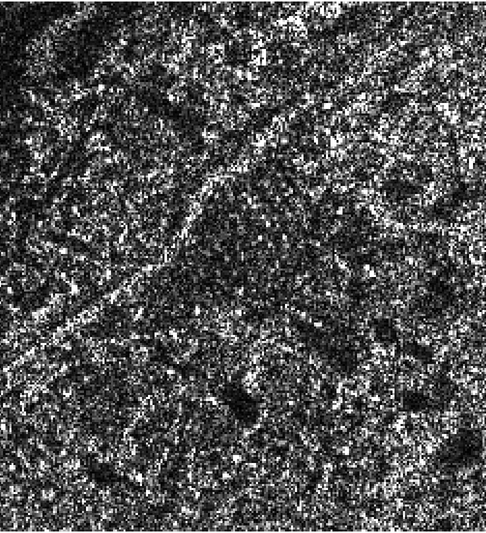


"×" wavelets - TempApp - db12

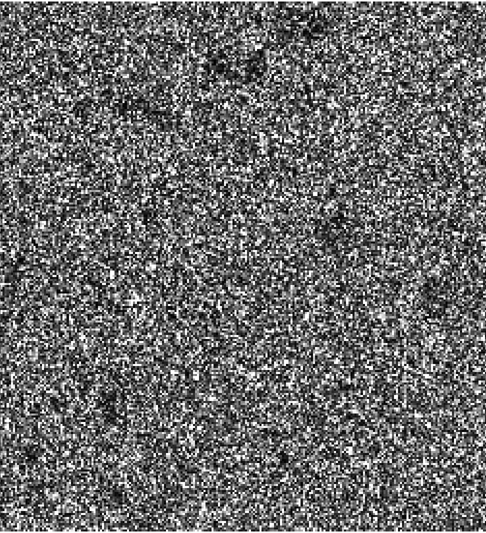


Which model (for sparsity)?

“$+$” wavelets - TempDet - ha12



“$\times$” wavelets - TempDet - ha12



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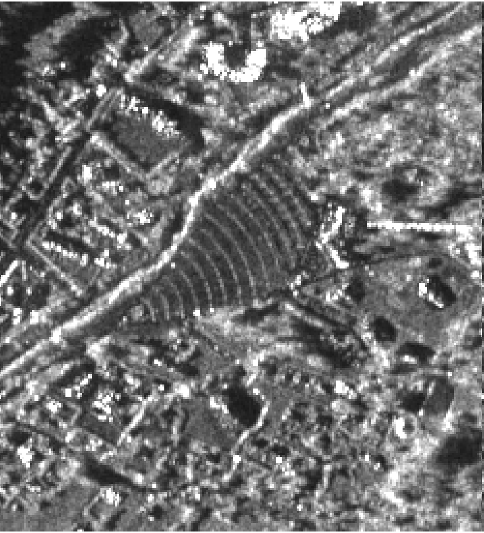

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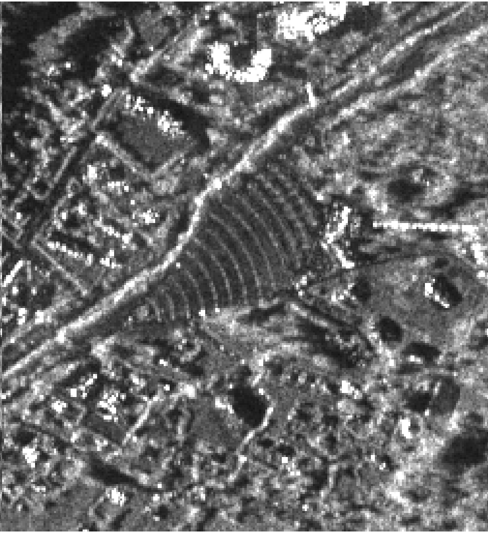

Geometric wavelets / Sparsity and noise

Geometric wavelets: sparsity and noise

Sparsity in a in multiplicative noise model means:

- shrinking to 1, the data that are close to 1 ...
- provided that noise is nice (in the geometric wavelet domain)!

$\Downarrow\Downarrow$

Statistical properties of geometric wavelet coefficients of speckle (noise model $(\star\star)$)?

Geometric wavelets / Stochasticity / Autocorrelation

Autocorrelation of geometric wavelet packet coefficients

Consider *N-length Haar type* approximation filter $\mathbf{h_0}$ and detail filter $\mathbf{h_1}$ given by

$$\mathbf{h_0}[k] = \nu \ \text{ for } \ k = 1, 2, \ldots, N. \tag{28}$$

$$\mathbf{h_1}[k] = (-1)^{k-1}\nu \ \text{ for } \ k = 1, 2, \ldots, N. \tag{29}$$

When using a sequence $\mathbf{h}_{\epsilon_1}, \mathbf{h}_{\epsilon_2}, \ldots, \mathbf{h}_{\epsilon_j}$ of such filters (wavelet packet subband $\mathbf{W}_{j,n}$), then the equivalent wavelet filter is

$$\left|\mathbf{H}_{j,n}(\omega)\right|^2 = 2^j \prod_{\ell=1}^{j} \left(\frac{\sin(2^{\ell-2}N\omega)}{\sin(2^{-1}(\omega + \epsilon_\ell \pi))} \right)^2 \tag{30}$$

and the autocorrelation $\mathrm{R}_{\mathbf{D}_{j,n}}$, where $\mathbf{D}_{j,n} = \log \mathbf{C}^{\times}_{j,n}[\mathbf{Y}]$ is:

$$\mathrm{R}_{\mathbf{D}_{j,n}}[m] = \frac{2^j}{\pi} \int_0^{\pi} \prod_{\ell=1}^{j} \left(\frac{\sin(2^{\ell-2}N\omega)}{\sin(2^{-1}(\omega + \epsilon_\ell \pi))} \right)^2 \gamma_{\mathbf{Y}}(\omega) \cos 2^j m\omega \, d\omega. \tag{31}$$

Geometric wavelets / Stochasticity / Autocorrelation

In the usual wavelet splitting scheme, only approximation coefficients are decomposed again (the shift parameter $n \in \{0, 1\}$). This implies filtering sequences with the form

$$\left(\underbrace{\mathbf{h}_0, \mathbf{h}_0, \ldots, \mathbf{h}_0}_{j \text{ times}}, \mathbf{h}_{\epsilon_{j+1}}\right)_{\epsilon_{j+1} \in \{0,1\}}$$

Consider a j-length approximation sequence $\left(\mathbf{h}_0^{\mathrm{Haar}}\right)_{\ell=1,2,\ldots,j}$ of standard Haar type ($N = 2$). Then, the equivalent filter of this sequence is:

$$\left|\mathbf{H}_{j,0}^{\mathrm{Haar}}(\omega)\right|^2 = 2^j \left(\frac{\mathrm{sinc}(2^{j-1}\omega)}{\mathrm{sinc}(2^{-1}\omega)}\right)^2, \qquad \text{where} \quad \mathrm{sinc}\omega = \sin\omega/\omega. \tag{32}$$

The autocorrelation $\mathrm{R}_{\mathbf{D}_{j,0}}^{\mathrm{Haar}}$ of the corresponding geometric wavelet coefficients is then:

$$\mathrm{R}_{\mathbf{D}_{j,0}}^{\mathrm{Haar}}[m] = \frac{2^j}{\pi} \int_0^{\pi} \left(\frac{\mathrm{sinc}(2^{j-1}\omega)}{\mathrm{sinc}(2^{-1}\omega)}\right)^2 \gamma_{\mathbf{Y}}(\omega) \cos 2^j m\omega \, d\omega \tag{33}$$

Limit Autocorrelation Function

$$\lim_{j \to +\infty} \mathrm{R}_{\mathbf{D}_{j,0}}^{\mathrm{Haar}}[m] = \gamma_{\mathbf{Y}}(0)\delta[m] \tag{34}$$

Geometric wavelets / Stochasticity / Distribution of REGG

The *Reciprocally Extended Generalized Gamma* (REGG) probability density function with scale parameter $\beta > 0$ and shape parameters κ, γ as:

$$f_{\kappa,\beta,\gamma}(x) = \frac{|\gamma|}{\beta\Gamma\left(\frac{\kappa}{\gamma}\right)} \left(\frac{x}{\beta}\right)^{\kappa-1} e^{-\left(\frac{x}{\beta}\right)^{\gamma}}, \quad \texttt{with} \quad \kappa\gamma > 0. \tag{35}$$

Function $f_{\kappa,\beta,\gamma}$

- is the probability density function of a Generalized Gamma random variable if $\kappa > 0$ and $\gamma > 0$ whereas it
- represents a Generalized Inverse Gamma random variable if $\kappa < 0$ and $\gamma < 0$.

Consider the random variable defined by

$$Z[k] = \prod_{\ell=0}^{N-1} \left(X[k-\ell]\right)^{\mathbf{h}[\ell]}$$

where $(X_k, X_{k-1}, \ldots, X_{k-N+1})$ are assumed to be independent and identically REGG distributed random variables with parameters (κ, β, γ).

Geometric wavelets / Stochasticity / Distribution of REGG

If $\mathbf{h} = \mathbf{h_0}$ is the Haar approximation filter of order N, then, the probability density function of Z is given by

$$f_Z(x) = \frac{|\gamma|/\nu}{{\beta_\nu}^N \Gamma^N\left(\frac{\kappa}{\gamma}\right)} G^{A_{\gamma,N}}\left(\left(\frac{{\beta_\nu}^N}{x}\right)^{\frac{|\gamma|}{\nu}} \middle| P_{\kappa,\gamma,\nu,N}\right) \quad (36)$$

where $\beta_\nu = \beta^\nu$, G is the Meijer function defined by

$$G\,{}^{m}_{p}\,{}^{n}_{q}\left(x \middle| \begin{array}{ccc} a_1 & \ldots & a_p \\ b_1 & \ldots & b_q \end{array}\right) = \frac{1}{2i\pi}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\prod_{\ell=1}^{m}\Gamma(b_\ell - s)\prod_{\ell=1}^{n}\Gamma(1 - a_\ell + s)}{\prod_{\ell=m+1}^{q}\Gamma(1 - b_\ell + s)\prod_{\ell=n+1}^{p}\Gamma(a_\ell - s)} x^s \mathrm{d}s,$$

with

$$A_{\gamma,N} = \begin{cases} \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} & \text{if } \gamma > 0 \\ \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} & \text{if } \gamma < 0 \end{cases}$$

$$P_{\kappa,\gamma,\nu,N} = \left(\underbrace{1 - \frac{\kappa-\nu}{\gamma}, 1 - \frac{\kappa-\nu}{\gamma}, \ldots, 1 - \frac{\kappa-\nu}{\gamma}}_{N \text{ times}}\right)$$

Geometric wavelets / Stochasticity / Distribution of REGG

If $\mathbf{h} = \mathbf{h_1}$ is the Haar detail filter of order N, then,

$$f_Z(x) = \frac{|\gamma|/\nu}{\beta_\nu{}^{*}\Gamma^N\left(\frac{\kappa}{\gamma}\right)} G^{B_{\gamma,N}}\left(\left(\frac{\beta_\nu{}^{*}}{x}\right)^{\frac{|\gamma|}{\nu}}\middle| P_{\kappa,\gamma,\nu,N}\right) \tag{37}$$

where $\beta_\nu{}^{*} = \beta^{(1-(-1)^N)/2\nu}$,

$$B_{\gamma,N} = \begin{cases} \begin{pmatrix} \lfloor\frac{N}{2}\rfloor & \lfloor\frac{N+1}{2}\rfloor \\ \lfloor\frac{N+1}{2}\rfloor & \lfloor\frac{N}{2}\rfloor \end{pmatrix} & \text{if } \gamma > 0 \\ \begin{pmatrix} \lfloor\frac{N+1}{2}\rfloor & \lfloor\frac{N}{2}\rfloor \\ \lfloor\frac{N}{2}\rfloor & \lfloor\frac{N+1}{2}\rfloor \end{pmatrix} & \text{if } \gamma < 0 \end{cases}$$

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Geometric wavelets / Stochasticity / Distribution of REGG

Geometric wavelet coefficients of REGG

- Highly non-Gaussian.
- Highly asymmetric.
- Heavy tails.

Geometric wavelet / Sparsity, Stochasticity and shrinkage

- Smooth shrinkage (for avoiding artifacts).
- Asymmetric shrinkage (with respect to the asymmetry of the distribution).
- Block shrinkage (for reducing the impact of the distribution tail).

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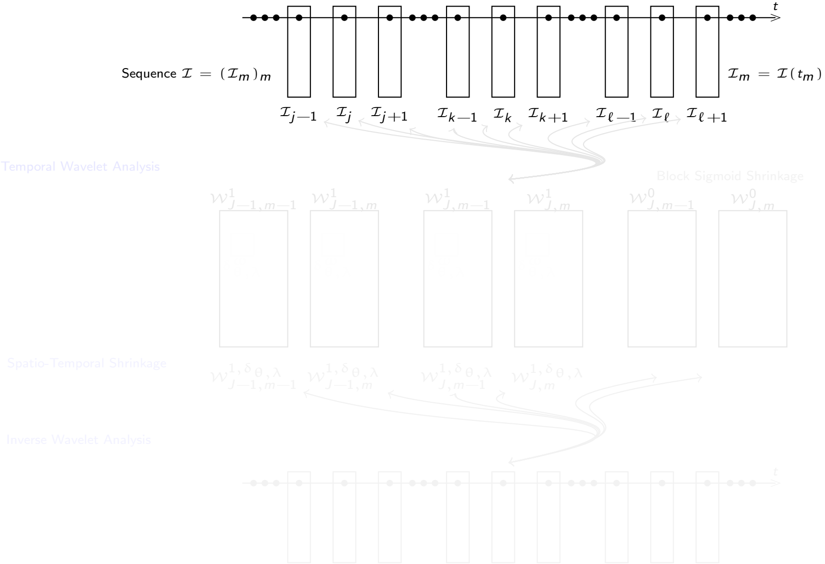
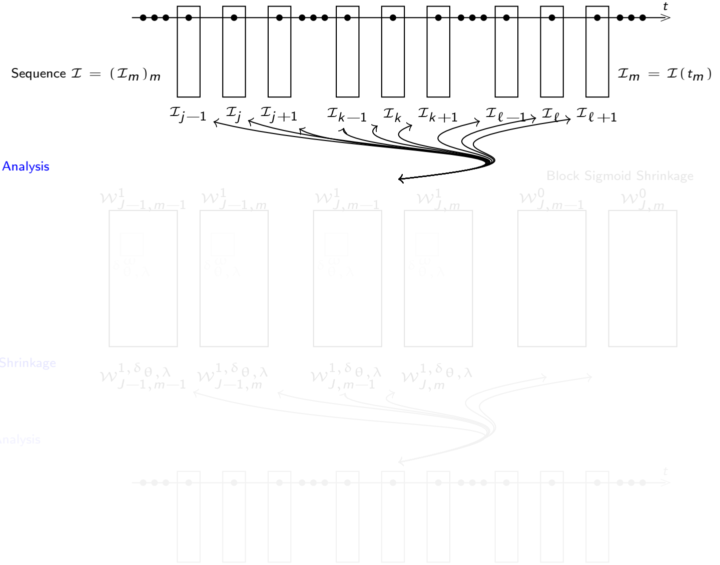
Method / Joint spatio-temporal filtering and change detection



Method / Joint spatio-temporal filtering and change detection

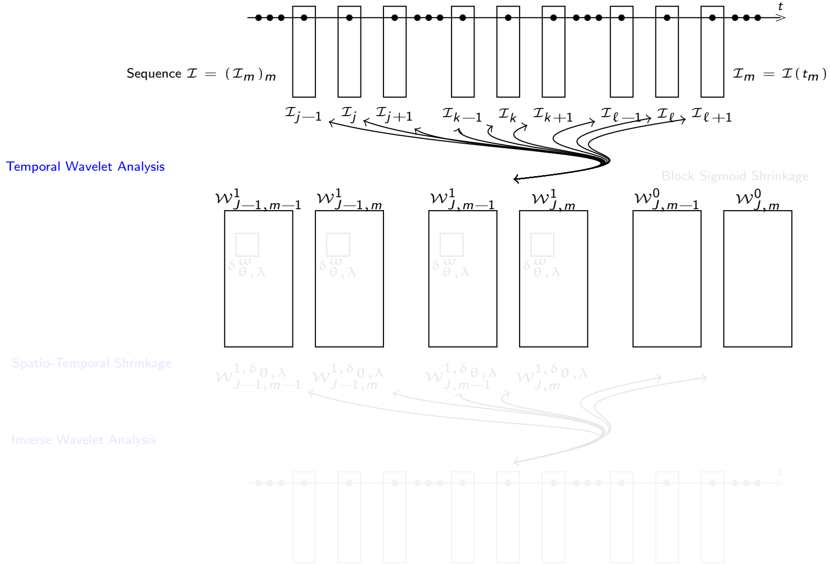


Temporal Wavelet Analysis

Spatio-Temporal Shrinkage

Inverse Wavelet Analysis

Method / Joint spatio-temporal filtering and change detection



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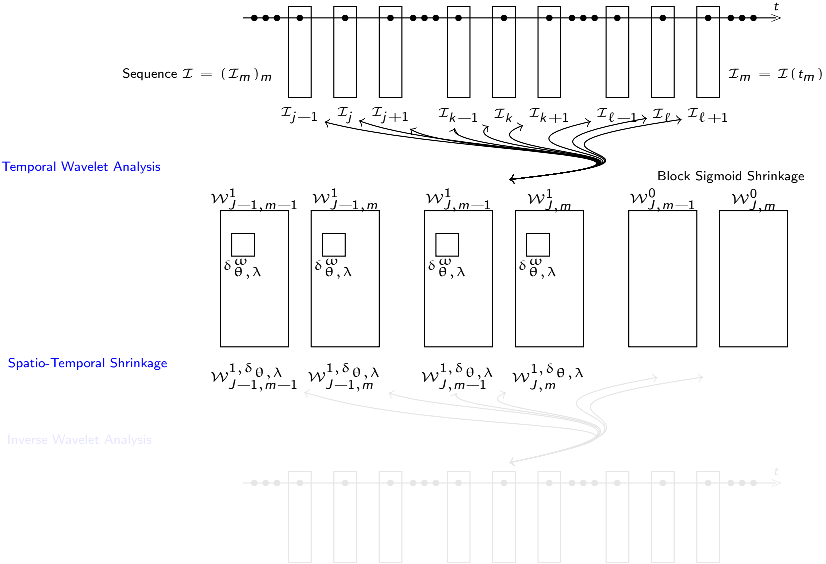

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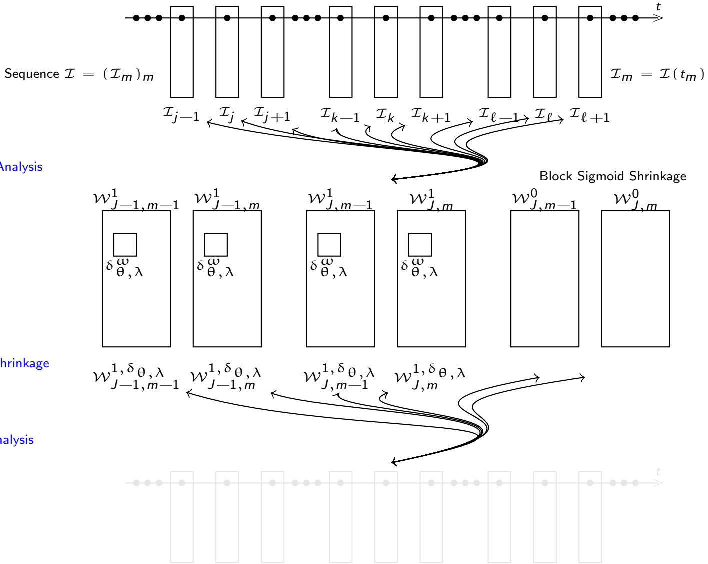

Method / Joint spatio-temporal filtering and change detection

Temporal Wavelet Analysis

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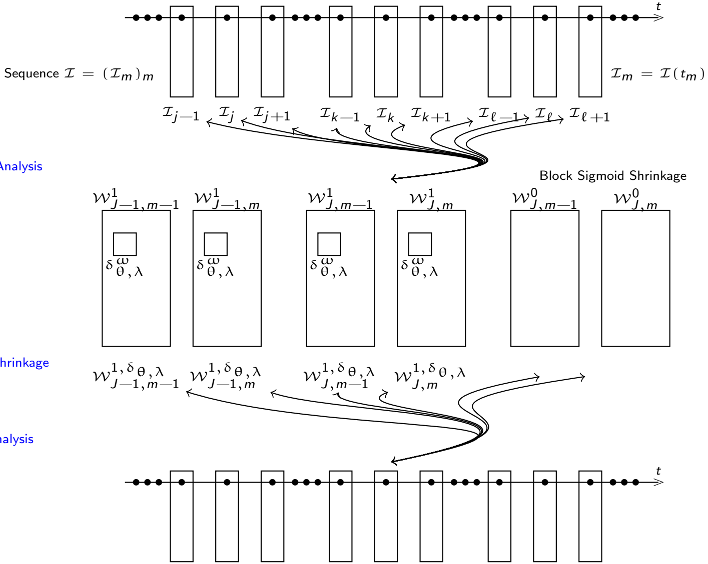


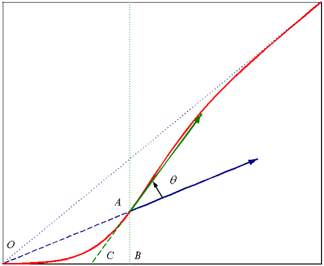
Geometric wavelets / Statistical properties REGG

Sigmoid shrinkage

$$\delta_{\theta_1,\theta_{-1},\lambda}(x) = \frac{x}{\left(1 + e^{-\frac{10 \sin \theta_x}{2 \cos \theta_x - \sin \theta_x}\left(\frac{|x|}{\lambda} - 1\right)}\right)}$$

where

$$\theta_x = \frac{1}{2}\left[\theta_1 + \theta_{-1} + \mathbf{sign}(x)\left(\theta_1 - \theta_{-1}\right)\right]$$



Spatio-temporal sigmoid shrinkage (spatial blocs on temporal differencing)

For a pixel $\mathrm{d}_{\mathcal{W}}\mathcal{I}_{m,n}(k)$ pertaining to a change-image, the sigmoid shrinkage:

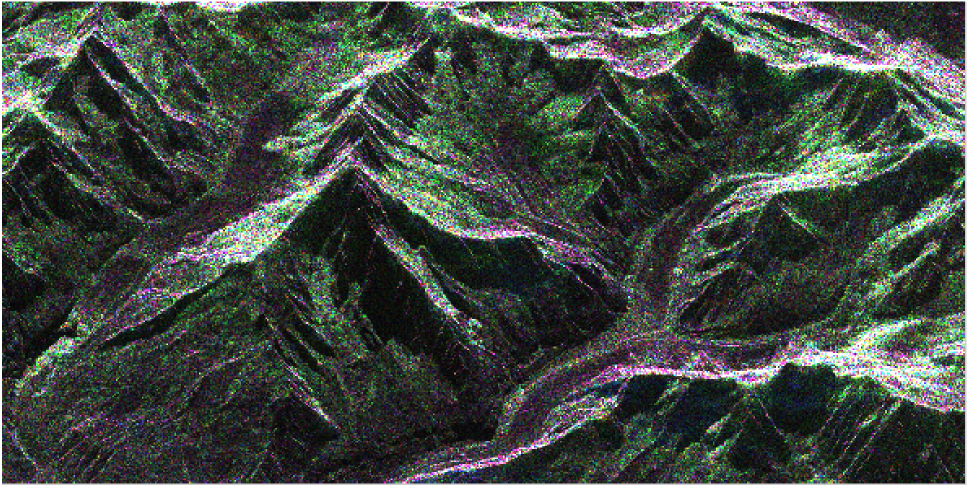
$$\delta_{\theta,\lambda}(\mathrm{d}_{\mathcal{W}}\mathcal{I}_{m,n}(k)) = \frac{\mathrm{d}_{\mathcal{W}}\mathcal{I}_{m,n}(k)}{1 + e^{-\frac{10 \sin \theta}{2 \cos \theta - \sin \theta}\left(\frac{||V_{\mathrm{d}_{\mathcal{W}}\mathcal{I}}(m,n,k)||_2}{\lambda} - 1'\right)}} \tag{38}$$

where

$$V_{\mathrm{d}_{\mathcal{W}}\mathcal{I}}(m,n,k) = \{\mathrm{d}_{\mathcal{W}}\mathcal{I}_{m,n}(k), m = m - \epsilon_0, \ldots, m + \epsilon_0, n = n - \nu_0, \ldots, n + n_0\}$$

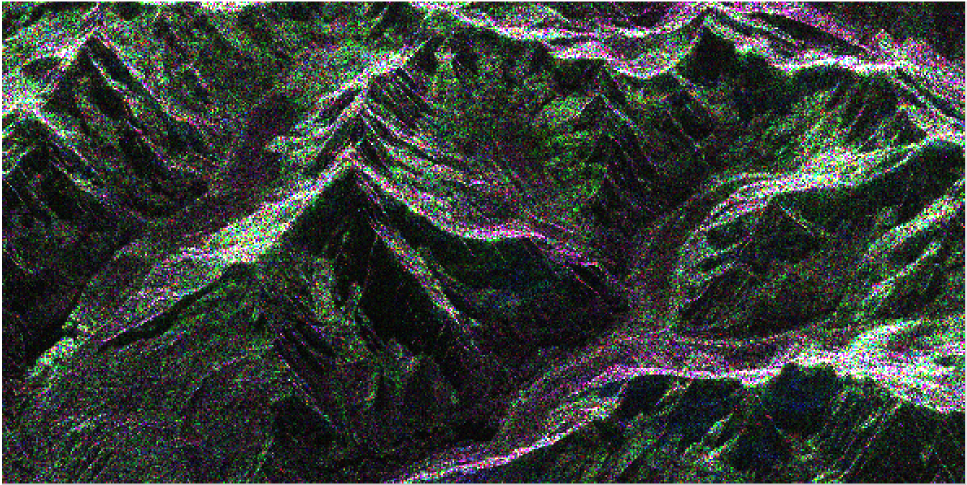
Method / Joint spatio-temporal filtering and change detection

$\mathcal{I}(t_1) \quad || \quad t_1 = 2009 - 02 - 22$



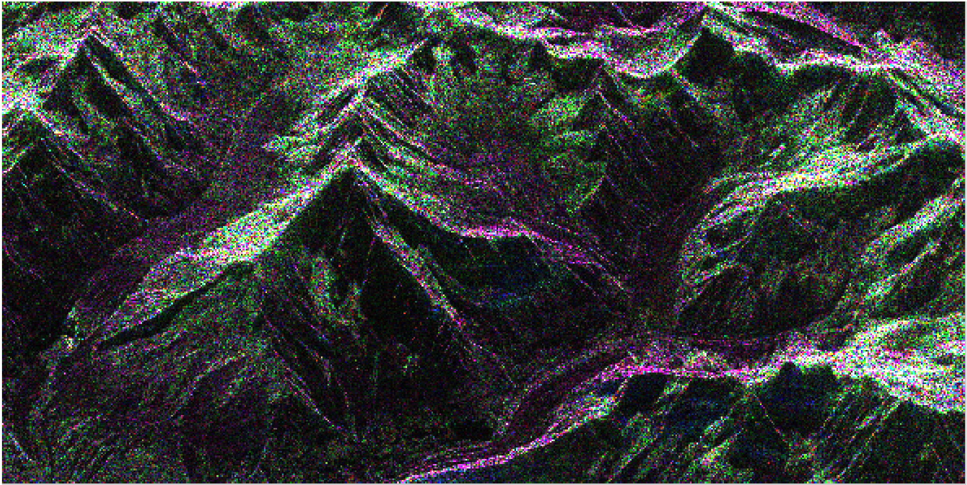
Method / Joint spatio-temporal filtering and change detection

$\mathcal{I}(t_2) \quad \| \quad t_2 = 2009 - 03 - 18$



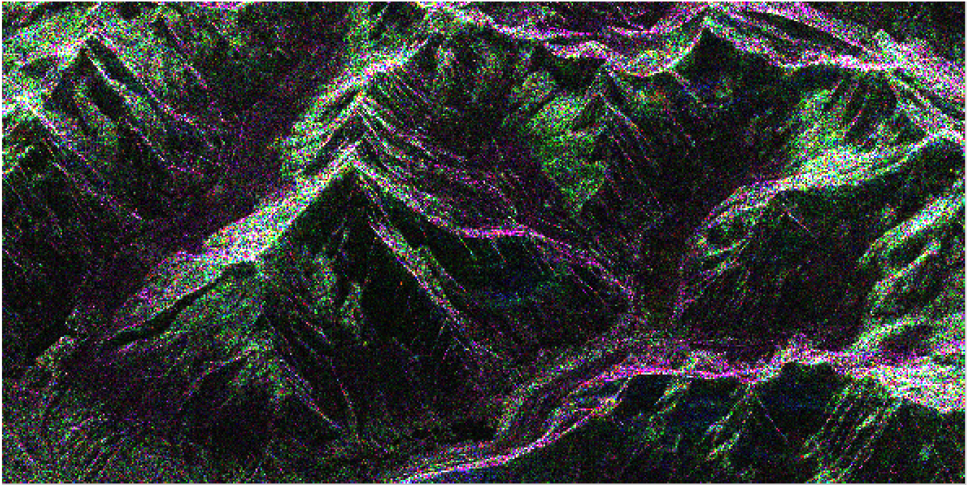
Method / Joint spatio-temporal filtering and change detection

$\mathcal{I}(t_3) \quad \| \quad t_3 = 2009 - 04 - 11$



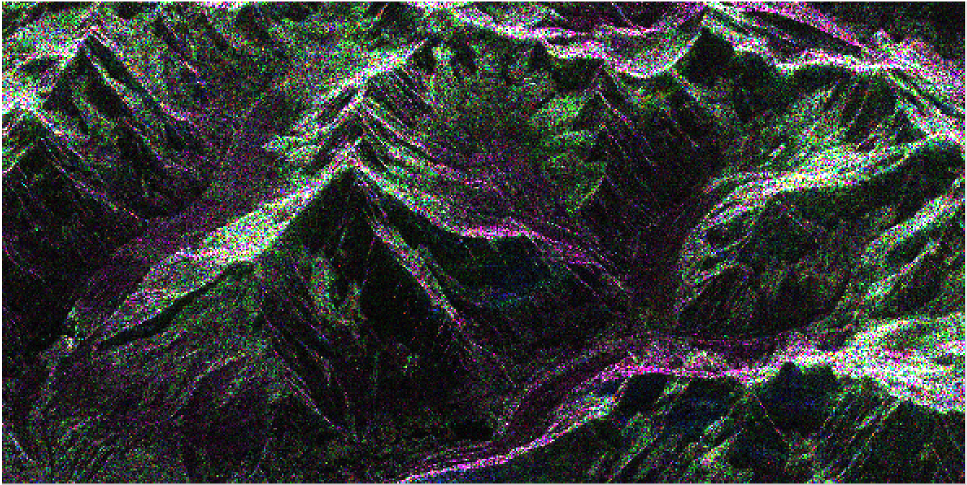
Method / Joint spatio-temporal filtering and change detection

$\mathcal{I}(t_4) \quad || \quad t_4 = 2009-05-05$



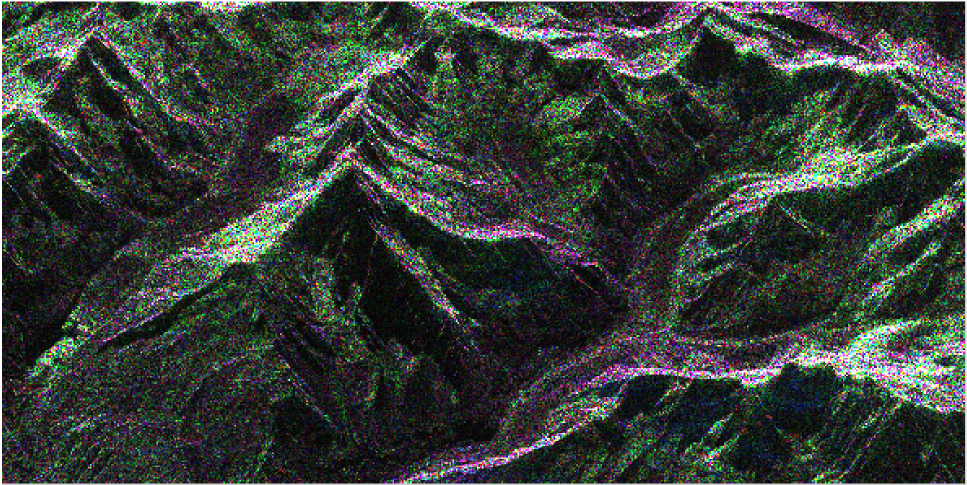
Method / Joint spatio-temporal filtering and change detection

$\mathcal{I}(t_3) \quad || \quad t_3 = 2009 - 04 - 11$



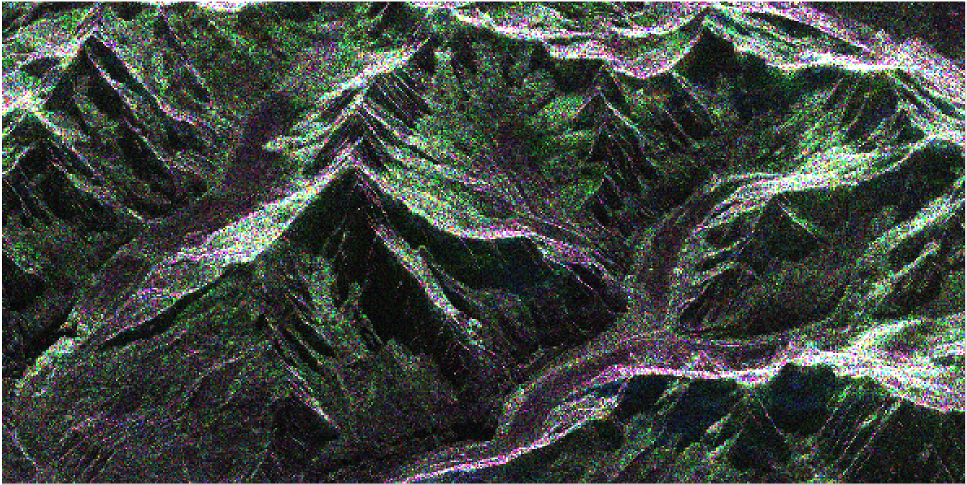
Method / Joint spatio-temporal filtering and change detection

$\mathcal{I}(t_2) \quad || \quad t_2 = 2009 - 03 - 18$



Method / Joint spatio-temporal filtering and change detection

$\mathcal{I}(t_1) \quad || \quad t_1 = 2009 - 02 - 22$

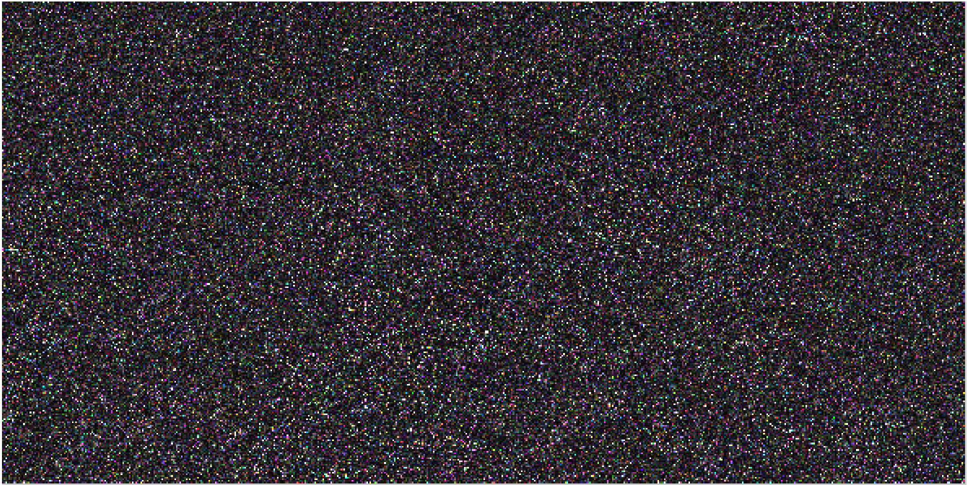


Method / Joint spatio-temporal filtering and change detection

$D_1[\mathcal{I}](1)$

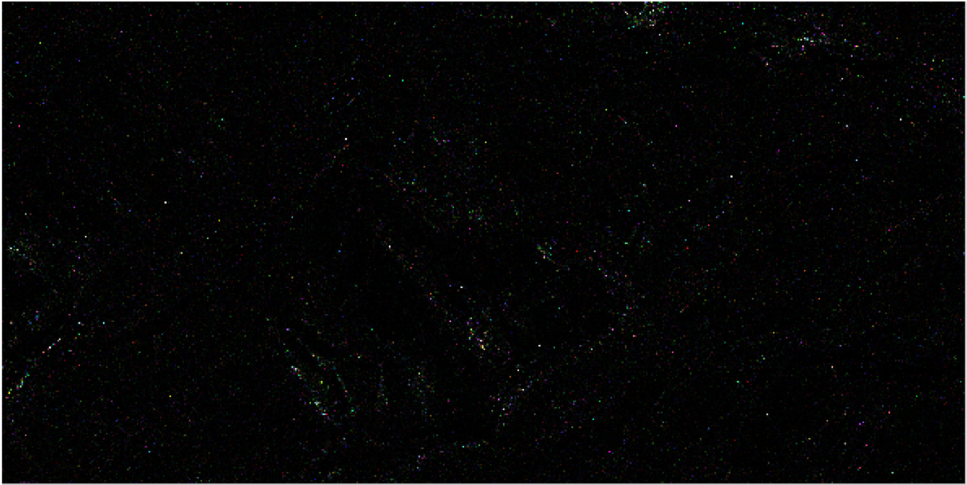
Method / Joint spatio-temporal filtering and change detection

$\mathcal{S}D_1[\mathcal{I}](1)$, Shrinkage λ_1



Method / Joint spatio-temporal filtering and change detection

$\mathcal{S}D_1[\mathcal{I}](1)$, Shrinkage λ_2

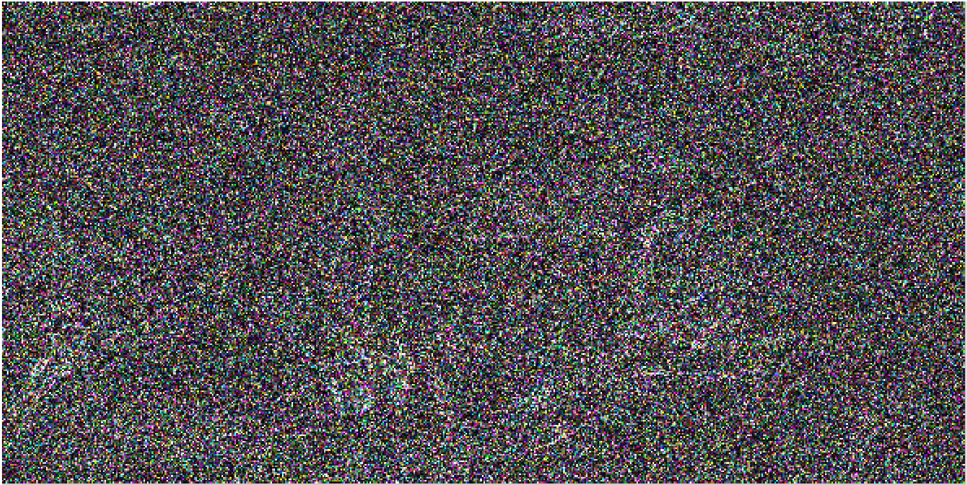


Method / Joint spatio-temporal filtering and change detection

$D_1[\mathcal{I}](2)$

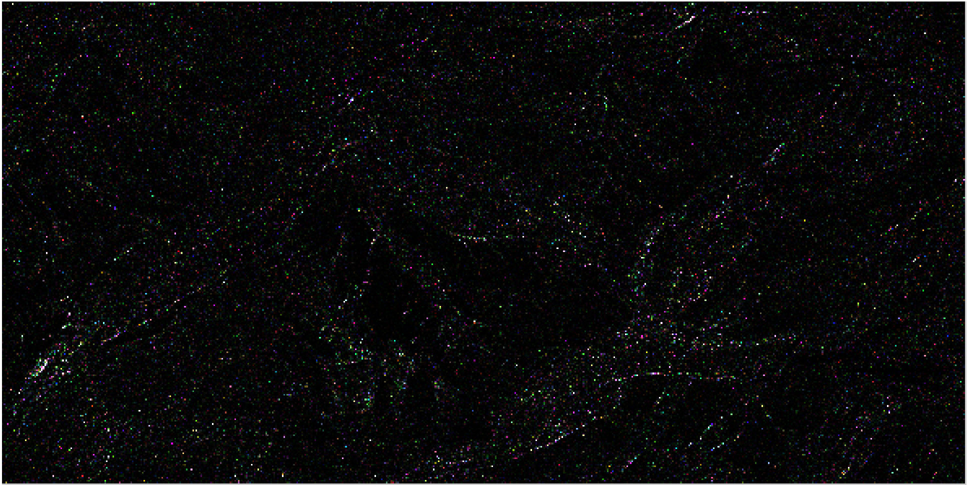
Method / Joint spatio-temporal filtering and change detection

$\mathcal{S}D_1[\mathcal{I}](2)$, Shrinkage λ_1



Method / Joint spatio-temporal filtering and change detection

$\mathcal{S}D_1[\mathcal{I}](2)$, Shrinkage λ_2

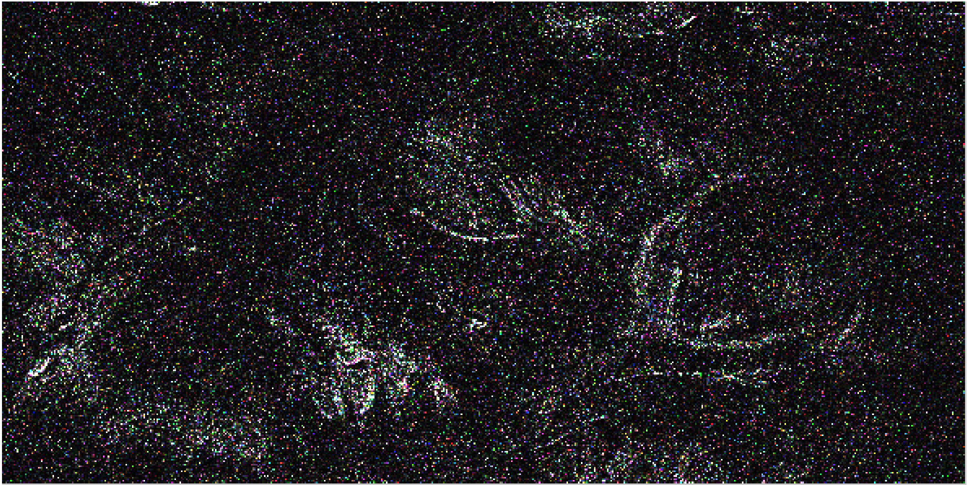


Method / Joint spatio-temporal filtering and change detection

$D_2[\mathcal{I}](1)$

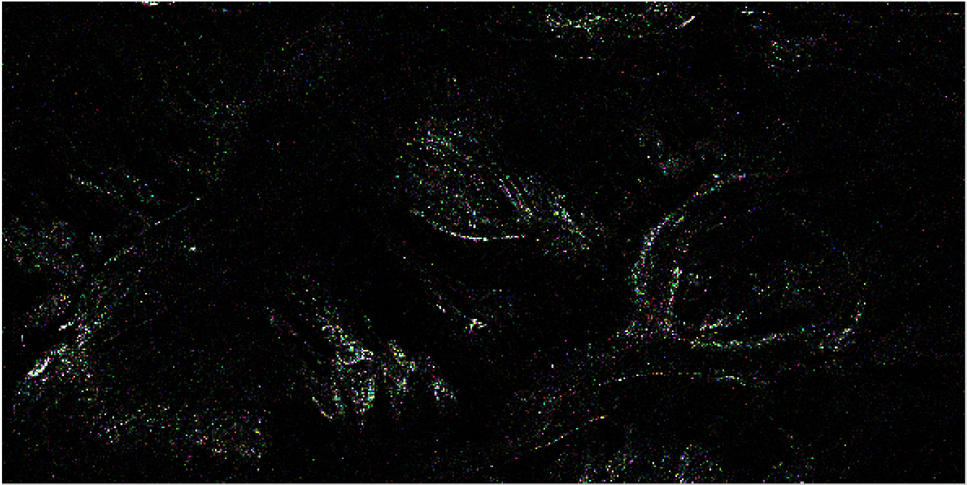
Method / Joint spatio-temporal filtering and change detection

$\mathcal{S}D_2[\mathcal{I}](1)$, Shrinkage λ_1



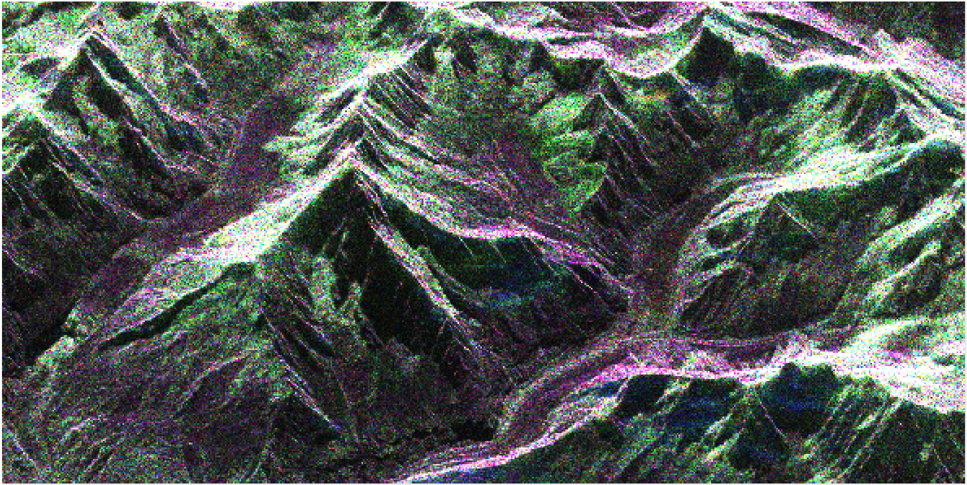
Method / Joint spatio-temporal filtering and change detection

$\mathcal{S}D_2[\mathcal{I}](1)$, Shrinkage λ_2



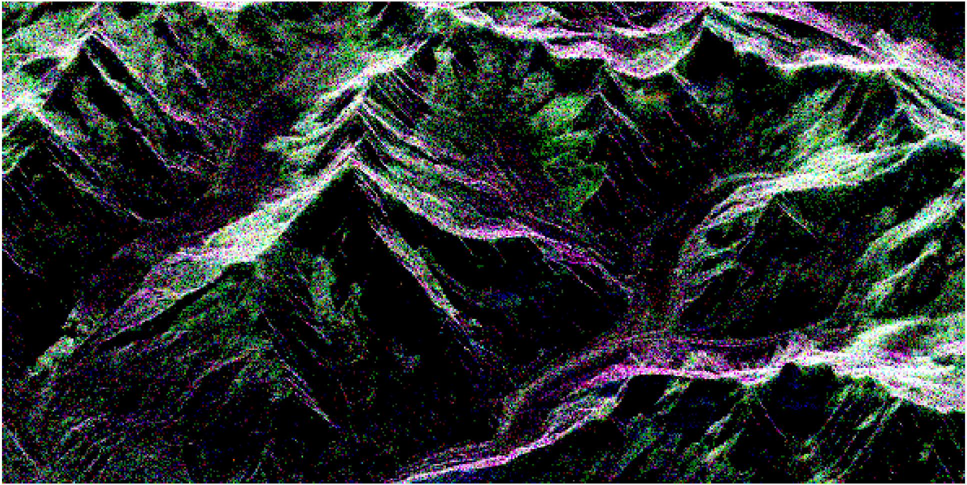
Method / Joint spatio-temporal filtering and change detection

$A_2[\mathcal{I}](1)$



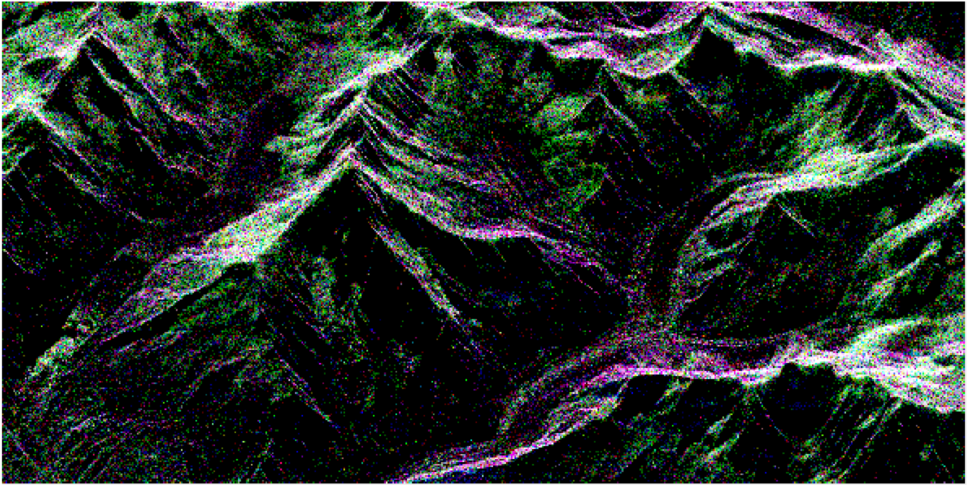
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_1

$$\Upsilon(t_1) \quad \| \quad t_1 = 2009 - 02 - 22$$



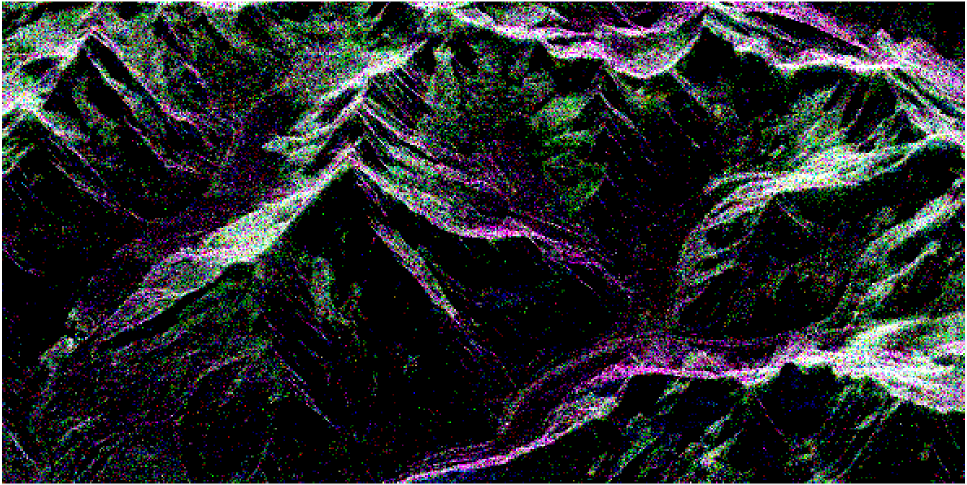
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_1

$$\Upsilon(t_2) \quad \| \quad t_2 = 2009 - 03 - 18$$



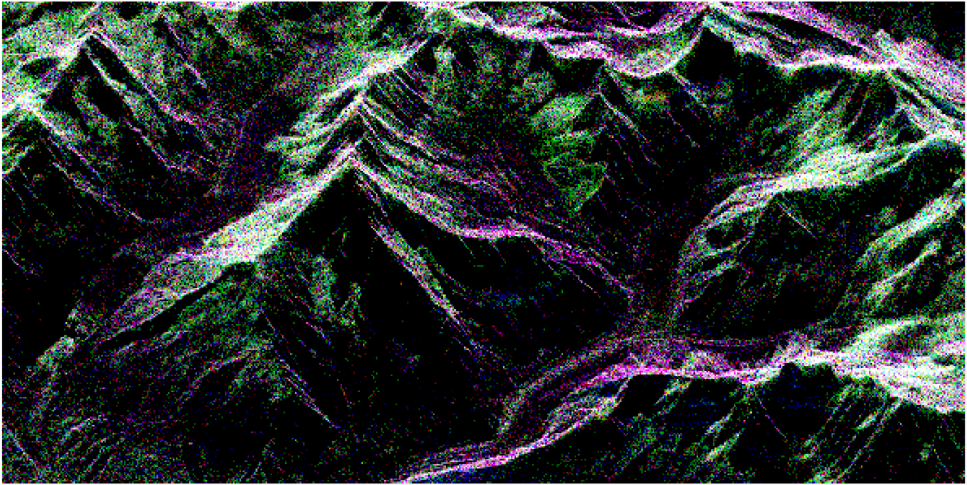
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_1

$$\Upsilon(t_3) \quad \| \quad t_3 = 2009 - 04 - 11$$



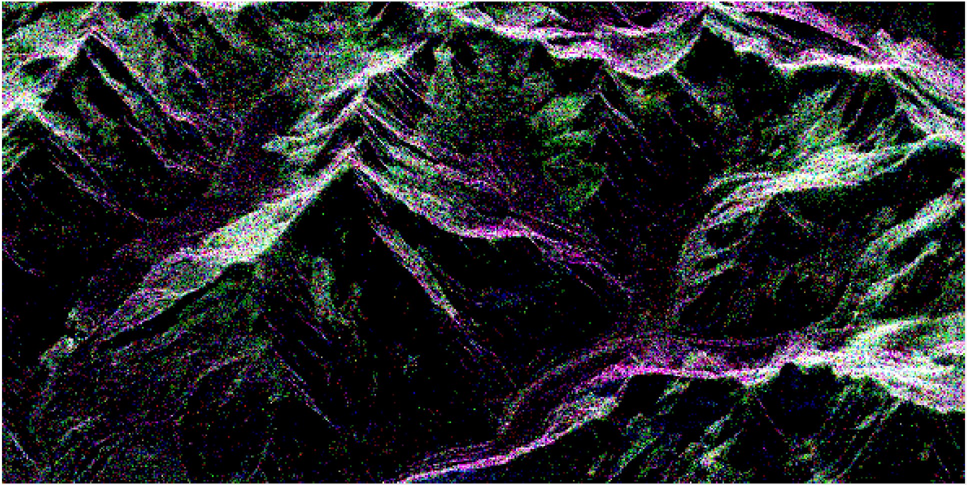
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_1

$\Upsilon(t_4) \quad || \quad t_4 = 2009 - 05 - 05$



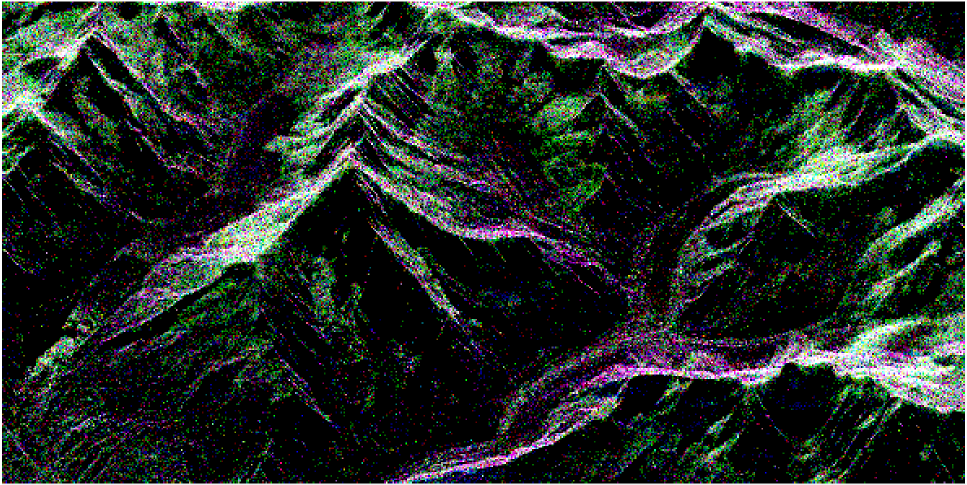
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_1

$\Upsilon(t_3) \quad \| \quad t_3 = 2009 - 04 - 11$



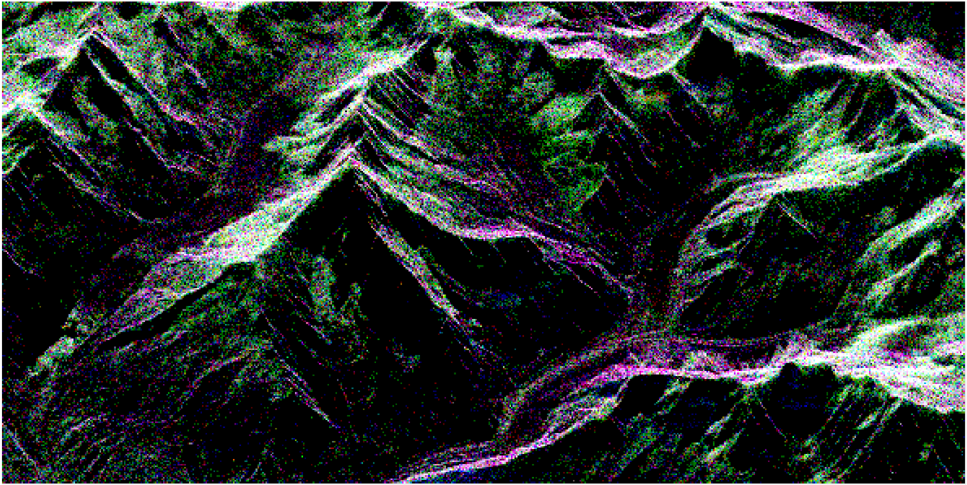
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_1

$\Upsilon(t_2) \quad || \quad t_2 = 2009 - 03 - 18$



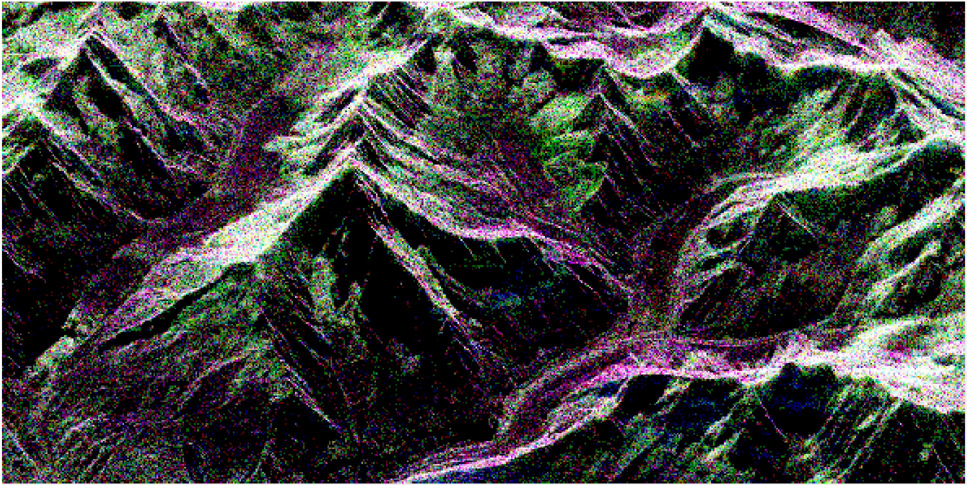
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_1

$$\Upsilon(t_1) \quad || \quad t_1 = 2009 - 02 - 22$$



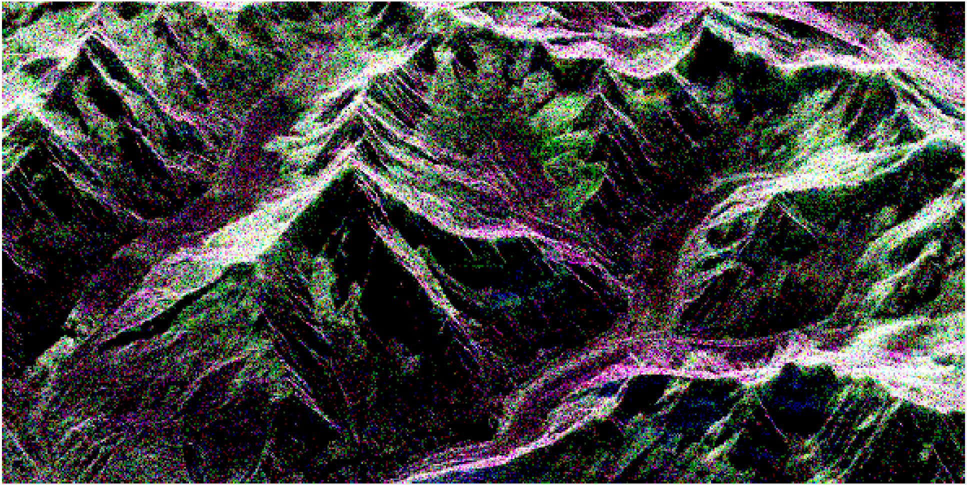
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_2

$\Upsilon(t_1) \quad || \quad t_1 = 2009 - 02 - 22$



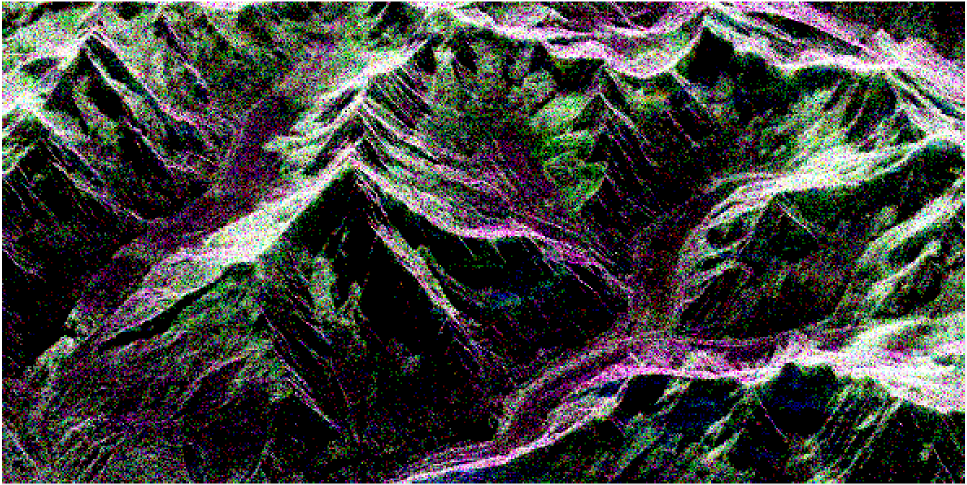
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_2

$$\Upsilon(t_2) \quad \| \quad t_2 = 2009 - 03 - 18$$



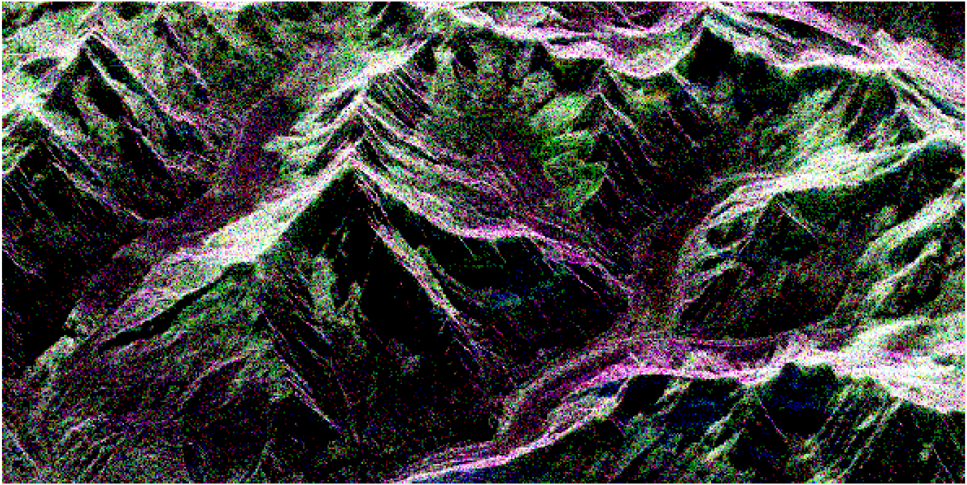
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_2

$$\Upsilon(t_3) \quad \| \quad t_3 = 2009 - 04 - 11$$



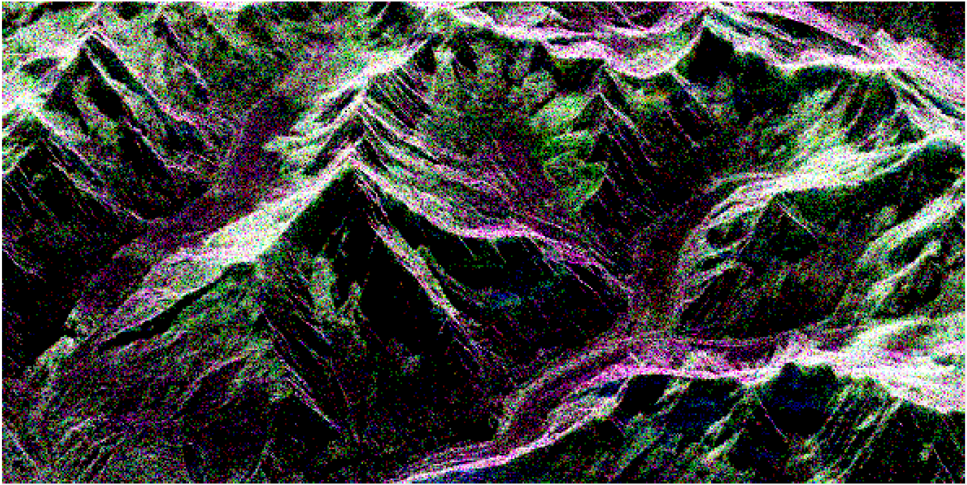
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_2

$\Upsilon(t_4) \quad \| \quad t_4 = 2009 - 05 - 05$



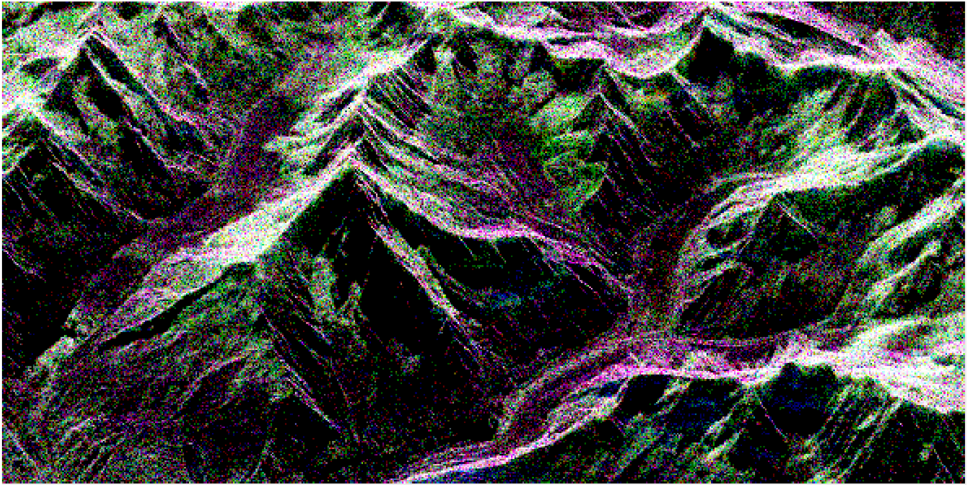
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_2

$\Upsilon(t_3) \quad || \quad t_3 = 2009 - 04 - 11$



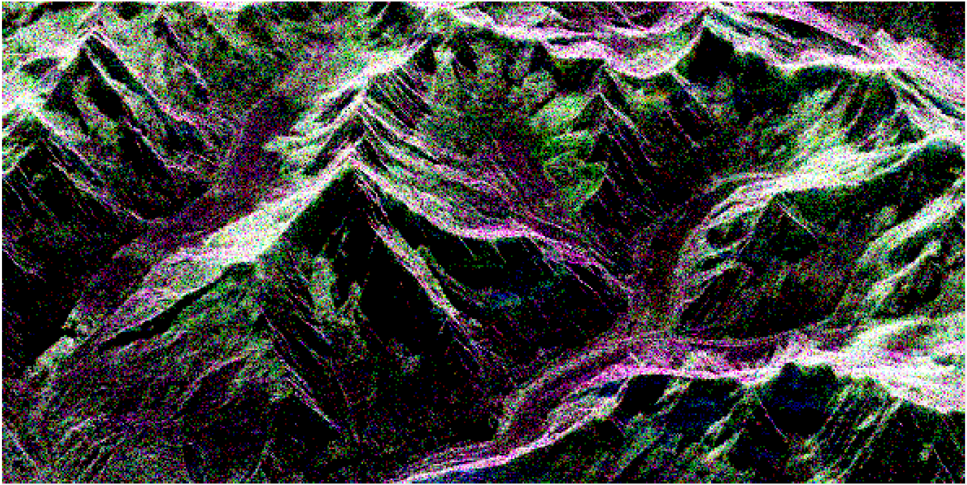
Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_2

$$\Upsilon(t_2) \quad \| \quad t_2 = 2009 - 03 - 18$$



Method / Joint spatio-temporal filtering and change detection / Shrinkage λ_2

$$\Upsilon(t_1) \quad || \quad t_1 = 2009 - 02 - 22$$



Outline

- SAR Model & Stationarity
- Geometric Wavelets and SAR
- SAR ITS Analysis
- Conclusion

Outline

Conclusion

- Analysis at two different levels:
  - $\Rightarrow$ Approximations / Temporal [Mean representatives of stable pixels/parts of the scene];
  - $\Rightarrow$ Details / Spatio-Temporal [change-images representatives of the scene dynamics].
- Workable for long time series of high spatial resolution + multichannel,
  - $\Rightarrow$ Wavelet on the temporal axis
  - $\Rightarrow$ Shrinkage with respect to spatio-temporal change information.
- Easy monitoring of the temporal evolution of Alps glaciers.

Prospects

- Limit distributions of REGG wavelets.
- Spatio-temporal wavelet variance analysis.
- Optimal parameters for sigmoid bloc shrinkage.
- Identifying stationary subsequences / seasonality.
- Compressive sensing in a geometric algebra.

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