ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Riesz and monogenic wavelet frames with applications to bioimaging

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Joint work with
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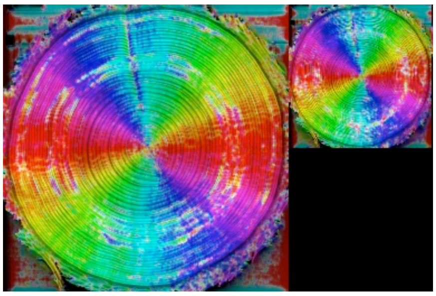


SIERRA, Grenoble, Octobre 11, 2012

Hilbert transform

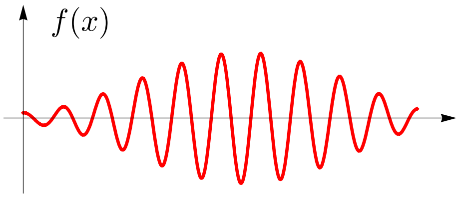
- Definition: $\mathcal{H}f(x) = (h*f)(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\mathrm{sgn}(\omega)\,\hat{f}(\omega) = -j\frac{\omega}{|\omega|}\hat{f}(\omega)$

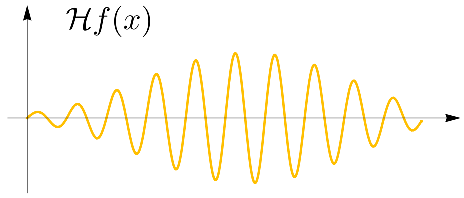
- Key properties
  - Shift invariance (LSI operator): $\mathcal{H}\{f(\cdot - x_0)\}(x) = \mathcal{H}f(x - x_0)$
  - Maps cosines into sines: $\mathcal{H}\{\cos(\omega_0 \cdot)\}(x) = \sin(\omega_0 x)$
  - Scale invariance: $\mathcal{H}\{f(\cdot/a)\}(x) = \mathcal{H}\{f(\cdot)\}(x/a)$
  - Unitary transform: $\forall \varphi_k, \varphi_l \in L_2(\mathbb{R}), \quad \langle \varphi_k, \varphi_l \rangle_{L_2} = \langle \mathcal{H}\varphi_k, \mathcal{H}\varphi_l \rangle_{L_2}$

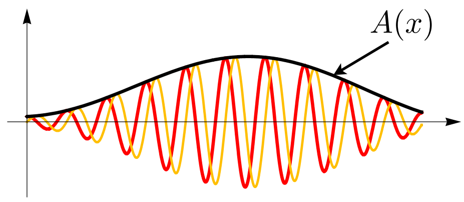
AM/FM signal analysis

- Analytical signal: $f_{\mathrm{ana}}(x) = f(x) + j\mathcal{H}f(x) = A(x)e^{j\xi(x)}$

- AM/FM analysis
  - Signal model: $f(x) = A(x)\cos(2\pi\nu(x)x + \xi_0)$ $\qquad$ ($A(x), \nu(x)$ slowly-varying)
  - Instantaneous modulus: $|A(x)| = |f_{\mathrm{ana}}(x)|$
  - Instantaneous frequency: $\nu(x) = \frac{\mathrm{d}\xi(x)}{\mathrm{d}x}$







Riesz transform

- Definition: $\boldsymbol{\mathcal{R}}f(\boldsymbol{x}) = \begin{pmatrix} \mathcal{R}_1 f(\boldsymbol{x}) \\ \vdots \\ \mathcal{R}_d f(\boldsymbol{x}) \end{pmatrix} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}\hat{f}(\boldsymbol{\omega})$

Multi-dimensional Fourier transform

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\mathrm{d}x_1\cdots\mathrm{d}x_d$$

with $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$

- Multi-channel convolution

$$\mathcal{R}_n f(\boldsymbol{x}) = (h_n * f)(\boldsymbol{x}) \quad \text{with} \quad h_n = \mathcal{R}_n\{\delta\} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\frac{\omega_n}{\|\boldsymbol{\omega}\|}$$

- Special case $d = 1$: the Hilbert transform

$$\mathcal{H}f(x) = (h * f)(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\mathrm{sgn}(\omega)\,\hat{f}(\omega) = -j\frac{\omega}{|\omega|}\hat{f}(\omega)$$

Riesz transform and derivatives

- Gradient

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = \left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \dots, \frac{\partial f(\boldsymbol{x})}{\partial x_d}\right) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad j\boldsymbol{\omega}\, \hat{f}(\boldsymbol{\omega})$$

- Fractional Laplacian

$$(-\Delta)^{\alpha} f(\boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \|\boldsymbol{\omega}\|^{2\alpha} \hat{f}(\boldsymbol{\omega})$$

Composition rule:

$$(-\Delta)^{\alpha_1}(-\Delta)^{\alpha_2} = (-\Delta)^{\alpha_1+\alpha_2}$$

with $(-\Delta)^0 =$ Identity

- Riesz transform and partial derivatives

$$\boldsymbol{\mathcal{R}} f(\boldsymbol{x}) = (-1)(-\Delta)^{-\frac{1}{2}} \boldsymbol{\nabla} f(\boldsymbol{x})$$

"Smoothed version of gradient"

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = -\boldsymbol{\mathcal{R}}(-\Delta)^{\frac{1}{2}} f(\boldsymbol{x})$$

Riesz transform in maths, SP and optics

- Riesz transform in mathematics
  - Fonctions conjuguées (Riesz 1920)
  - Singular integral operators (Calderon-Zygmund, 1955; Stein, 1970)
- Hilbert and Riesz transform in signal processing
  - Analytical signal (Gabor, 1946; Ville 1948)
  - 2D extension: Monogenic signal analysis (Felsberg, 2001)
  - Phased-based feature detection (Noble-Brady *et al.*, 2004)
- Riesz transform in optics
  - Radial Hilbert transform (Davis, 2000)
  - Spiral phase quadrature transform (Larkin, 2001)

Steerability and directional Hilbert transform

- Directional Hilbert transform

Unit vector: $\boldsymbol{u} = (u_1, \cdots, u_d)$

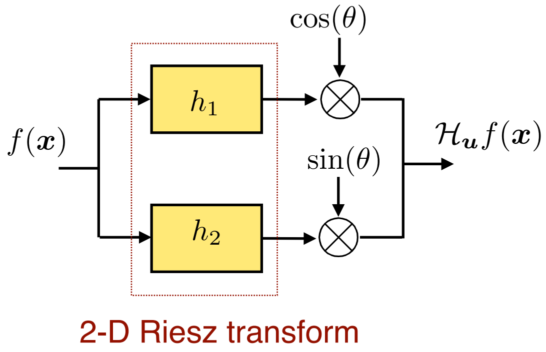
$$\mathcal{H}_{\boldsymbol{u}} f(\boldsymbol{x}) = \sum_{n=1}^{d} u_n \mathcal{R}_n f(\boldsymbol{x}) = \langle \boldsymbol{u}, \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) \rangle$$

Hilbert-like behavior in direction $\boldsymbol{u}$: $\widehat{\mathcal{H}_{\boldsymbol{u}}}(\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\omega\boldsymbol{u}} = -j\mathrm{sgn}(\omega)$

- Implementation in 2-D

$\boldsymbol{u} = (\cos\theta, \sin\theta)$

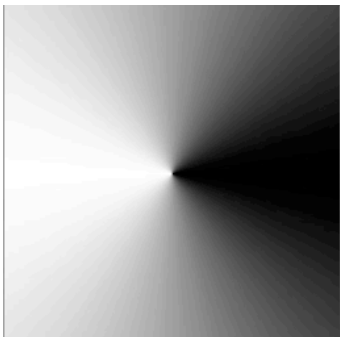
Gradient-like steerable filterbank



Visualization in the frequency domain

- Frequency response of directional Hilbert transform

$$\widehat{\mathcal{H}_{\boldsymbol{u}}}(\boldsymbol{\omega}) = \langle \boldsymbol{u}, -j\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \rangle$$



$$\mathcal{H}_{\boldsymbol{u}} f(\boldsymbol{x}) = (-1)(-\Delta)^{-\frac{1}{2}} \mathrm{D}_{\boldsymbol{u}} f(\boldsymbol{x})$$

“Smoothed version of directional derivative”

Properties of the Riesz transform

- Shift invariance: $\forall \boldsymbol{x}_0 \in \mathbb{R}^d, \quad \boldsymbol{\mathcal{R}}\{f(\cdot - \boldsymbol{x}_0)\}(\boldsymbol{x}) = \boldsymbol{\mathcal{R}}\{f(\cdot)\}(\boldsymbol{x} - \boldsymbol{x}_0)$

- Scale invariance: $\forall a \in \mathbb{R}^+, \quad \boldsymbol{\mathcal{R}}\{f(\cdot/a)\}(\boldsymbol{x}) = \boldsymbol{\mathcal{R}}\{f(\cdot)\}(\boldsymbol{x}/a)$

Maps wavelets into gradient-like wavelets:

$$\boldsymbol{\mathcal{R}}\left\{\psi\left(\frac{\cdot - \boldsymbol{x}_0}{a}\right)\right\}(\boldsymbol{x}) = \nabla\{\phi\}\left(\frac{\cdot - \boldsymbol{x}_0}{a}\right) \quad \text{with} \quad \phi = \mathcal{F}^{-1}\left\{j\frac{\hat{\psi}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|}\right\}$$

- Adjoint operator

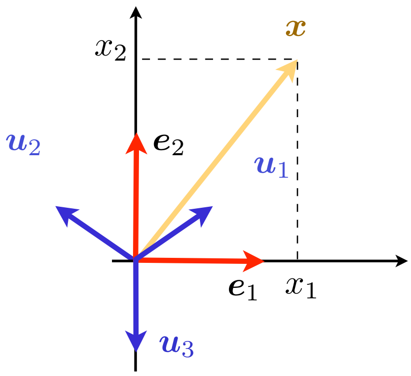
$$\boldsymbol{\mathcal{R}}^*\boldsymbol{r}(\boldsymbol{x}) = \mathcal{R}_1^* r_1(\boldsymbol{x}) + \cdots + \mathcal{R}_d^* r_d(\boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad j\frac{\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{r}}(\boldsymbol{\omega})$$

- Self-reversibility

$$\forall f \in L_2(\mathbb{R}^d), \quad \boldsymbol{\mathcal{R}}^*\boldsymbol{\mathcal{R}}f(\boldsymbol{x}) = \sum_{n=1}^{d} \mathcal{R}_n^*\mathcal{R}_n f(\boldsymbol{x}) = f(\boldsymbol{x})$$

CONTENT

Frame = redundant extension of a basis



$$\forall \boldsymbol{x} \in \mathbb{R}^2, \quad \boldsymbol{x} = \langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 + \langle \boldsymbol{x}, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2$$

$$\boldsymbol{x} = \langle \boldsymbol{x}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle \boldsymbol{x}, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 + \langle \boldsymbol{x}, \boldsymbol{u}_3 \rangle \boldsymbol{u}_3$$

- Definition
  - A family of functions $\{\psi_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$ is called a tight/Parseval frame of $L_2(\mathbb{R}^d)$ iff.

$$\forall f \in L_2(\mathbb{R}^d), \quad \|f\|_{L_2}^2 = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\langle \psi_{\boldsymbol{k}}, f \rangle_{L_2}|^2$$

- Analysis/synthesis formula
  - $\forall f \in L_2(\mathbb{R}^d), \quad f = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \langle \psi_{\boldsymbol{k}}, f \rangle_{L_2} \psi_{\boldsymbol{k}}$

Design of a primal isotropic wavelet

- Frequency domain design of bandlimited wavelets

Theorem: Let $\hat{\psi}(\boldsymbol{\omega}) = h(\|\boldsymbol{\omega}\|) = h(\omega)$ with

Condition (1) : $h(\omega) = 0, \forall \omega > \pi$ (Bandlimited)

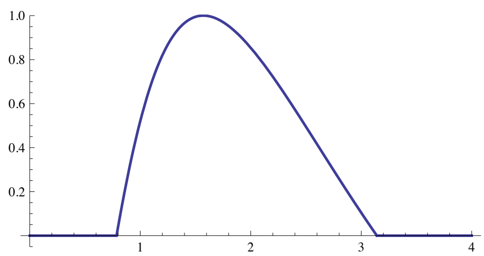
Condition (2) : $\sum_{i \in \mathbb{Z}} |h(2^i \omega)|^2 = 1$ (Self-reversibility)

Condition (3) : $\left.\frac{\mathrm{d}^n h(\omega)}{\mathrm{d}\omega^n}\right|_{\omega=0} = 0,$ for $n = 0, \ldots, N$ (Vanishing moments)

Then the isotropic wavelet $\psi = \mathcal{F}^{-1}\{\hat{\psi}\}$ generates a tight wavelet frame of $L_2(\mathbb{R}^d)$.

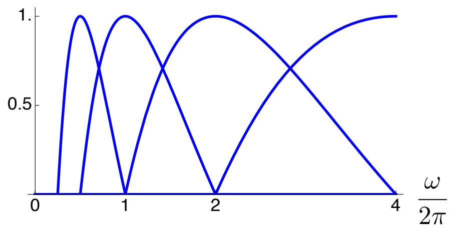
- Simoncelli's "log-Gabor" solution

$$h(\omega) = \begin{cases} \cos\left(\frac{\pi}{2} \log_2\left(\frac{2\omega}{\pi}\right)\right), & \frac{\pi}{4} < \|\omega\| \leq \pi \\ 0, & \text{otherwise} \end{cases}$$



Backbone: primal isotropic wavelet pyramid

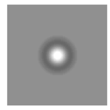
Radial wavelet filters



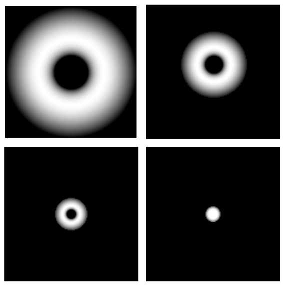
Tight frame property:

$$\sum_{i\in\mathbb{Z}} |\hat{\psi}(\boldsymbol{\omega}/2^i)|^2 = 1$$

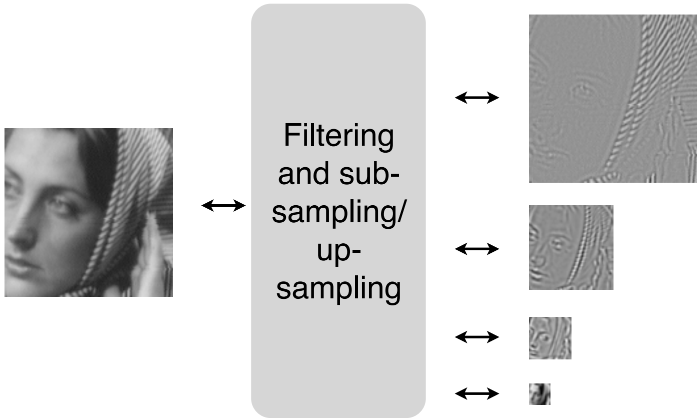
$$\psi(\boldsymbol{x}) = \psi(\|\boldsymbol{x}\|)$$



2D frequency view



Wavelet coefficients



Perfect isotropy

no preferred direction

Construction of tight, gradient-like wavelet frame

- Primary tight wavelet frame of $L_2(\mathbb{R}^d)$

$$\forall f \in L_2(\mathbb{R}^d), \quad f(\boldsymbol{x}) = \sum_{i\in\mathbb{Z}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle f, \psi_{i,\boldsymbol{k}} \rangle_{L_2}\, \psi_{i,\boldsymbol{k}}(\boldsymbol{x})$$

Wavelet property: $\psi_{i,\boldsymbol{k}}(\boldsymbol{x}) = 2^{-\frac{id}{2}} \psi\left(\frac{\boldsymbol{x}-2^i\boldsymbol{k}}{2^i}\right)$

Proposition
Let $\{\psi_{i,\boldsymbol{k}}\}$ be a primal tight wavelet frame of $L_2(\mathbb{R}^d)$. Then, $\{\boldsymbol{\mathcal{R}}\psi_{i,\boldsymbol{k}} = \nabla\phi_{i,\boldsymbol{k}}\}$ is a gradient-like tight wavelet frame such that

$$\forall f \in L_2(\mathbb{R}^d), \quad f(\boldsymbol{x}) = \sum_{i\in\mathbb{Z}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathbf{w}_{i,\boldsymbol{k}}^T \boldsymbol{\mathcal{R}}\psi_{i,\boldsymbol{k}}(\boldsymbol{x}) \ \text{ with } \ \mathbf{w}_{i,\boldsymbol{k}} = \langle f, \boldsymbol{\mathcal{R}}\psi_{i,\boldsymbol{k}} \rangle_{L_2}$$

Proof:

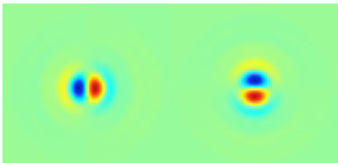
Use self-reversibility: $f(\boldsymbol{x}) = \sum_{n=1}^{d} \mathcal{R}_n \mathcal{R}_n^* f(\boldsymbol{x}) = f(\boldsymbol{x})$

with $\mathcal{R}_n^* f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \sum_{i\in\mathbb{Z}} \langle \mathcal{R}_n^* f, \psi_{i,\boldsymbol{k}} \rangle \psi_{i,\boldsymbol{k}}$

Gradient-like Riesz wavelets (self-reversible)

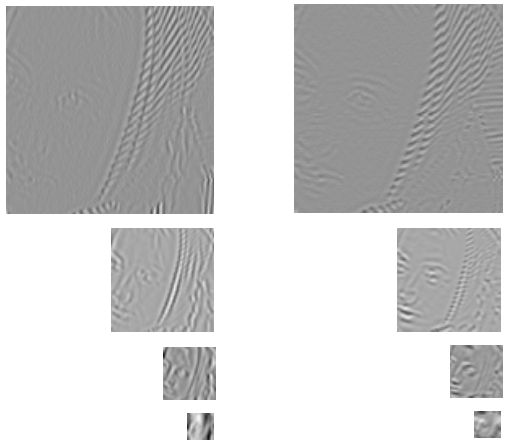
Steerable Riesz wavelets

$\cos\theta$ $\sin\theta$



$$\left(\mathcal{R}_1\psi = \frac{\partial\phi_1}{\partial x_1}, \quad \mathcal{R}_2\psi = \frac{\partial\phi_1}{\partial x_2}\right)$$

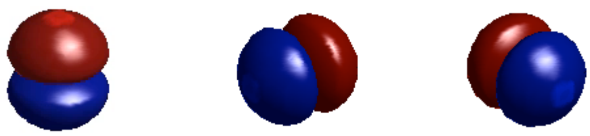
Wavelet coefficients



multi-scale x-derivative

multiscale y-derivative

3-D version (iso-surface display)



Application 1: multi-scale structure analysis

- Gadient-like wavelet transform

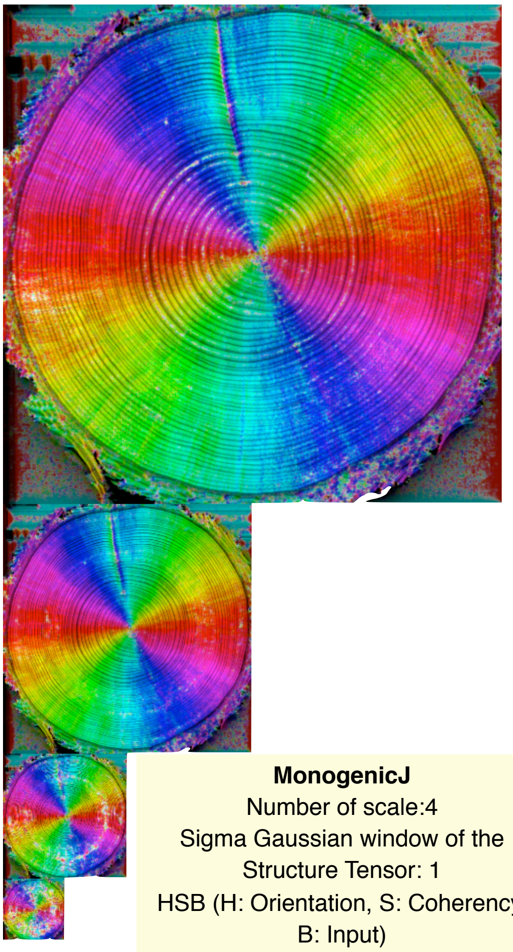
$$\mathbf{w}_i[\boldsymbol{k}] = \langle f, \boldsymbol{\mathcal{R}}\psi_{i,\boldsymbol{k}}\rangle = \langle f, \boldsymbol{\nabla}\phi_{i,\boldsymbol{k}}\rangle$$

- Wavelet-domain structure tensor (symmetric $d \times d$ matrix)

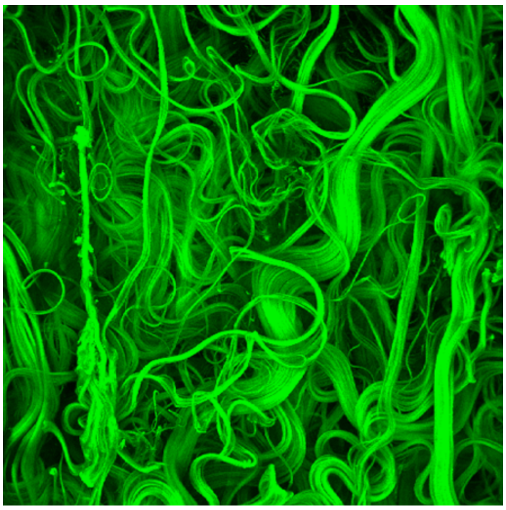
$$J_i[\boldsymbol{k}] = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} e^{-\frac{\|\boldsymbol{n}\|^2}{2\sigma^2}} \mathbf{w}_i[\boldsymbol{k}+\boldsymbol{n}]\mathbf{w}_i^T[\boldsymbol{k}+\boldsymbol{n}]$$

- Local features
  - Local wavelet energy: $E = \mathrm{trace}(\mathbf{J}) = \sum_{n=1}^{d} \lambda_n$
  - Maximum directional energy: λ_1 (maximum eigenvalue of $\mathbf{J}$)
  - Orientation: $\mathbf{u}_1$ (eigenvector associated with λ_1)
  - Coherency: $0 \leq C = \dfrac{\lambda_1 - \lambda_{\min}}{\lambda_1 + \lambda_{\min}}$

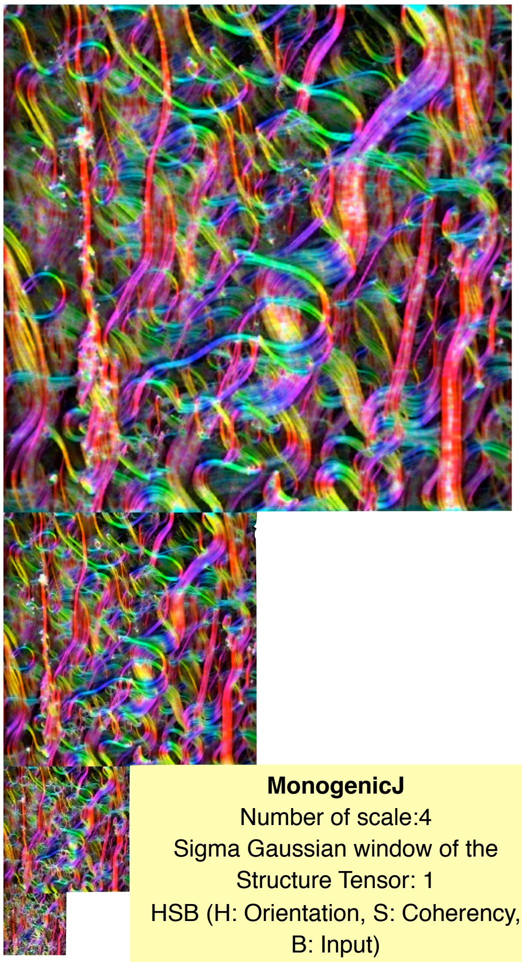
Dendrochronology image



MonogenicJ
Number of scale:4
Sigma Gaussian window of the Structure Tensor: 1
HSB (H: Orientation, S: Coherency, B: Input)

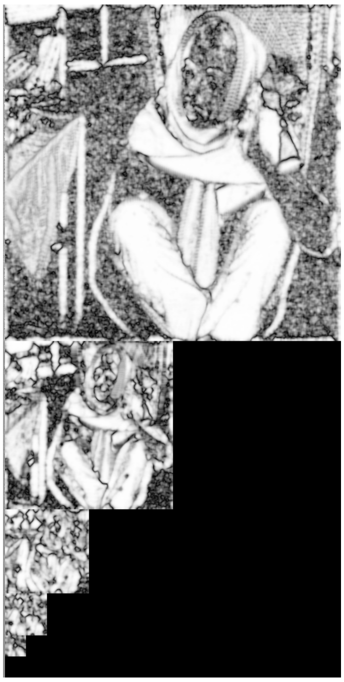


Collagen fiber in adventitia of rabbit carotids *ex vivo*. 3D confocal microscopy. Maximum Intensity Projection.
Rana Rezakhaniha, LHCT, EPFL

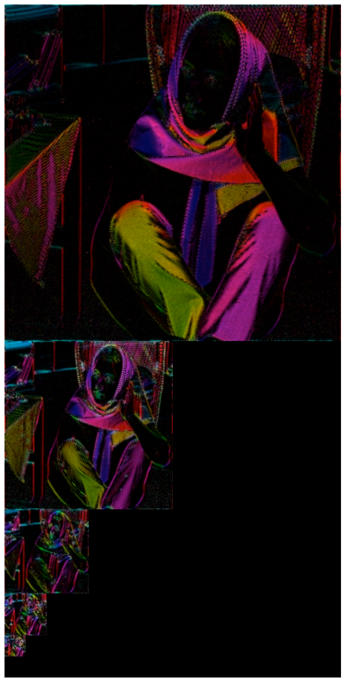


MonogenicJ
Number of scale:4
Sigma Gaussian window of the Structure Tensor: 1
HSB (H: Orientation, S: Coherency, B: Input)

Example: Coherence analysis of Barbara



Coherency



HSB
Hue: Orientation
Saturation: Coherency
Brightness: Modulus

Example: directional analysis of filaments in 3-D

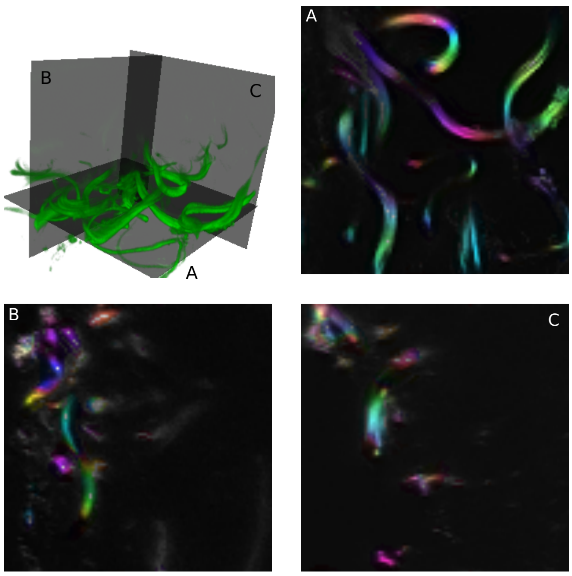
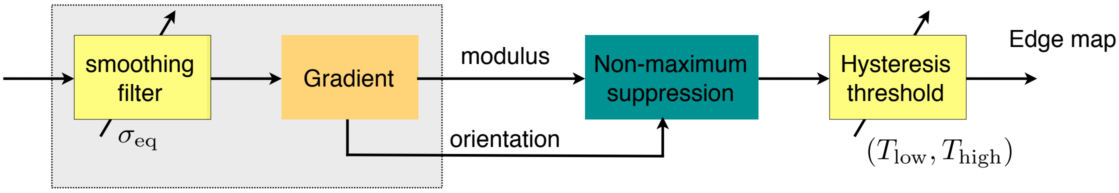


Fig. 3. Top-Left: a 3D image stack of collagen filaments. From top-right to bottom-right: color encoded (hue) monogenic direction at scale 2 for the plans A, B, and C, after an orthogonal projection onto the xy, xz and yz planes, respectively. The saturation indicates the value of the local coherency.

Application 2: Multiscale edge detection

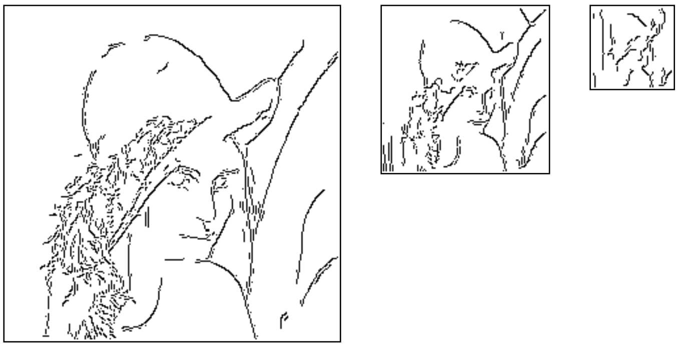
- Canny's state-of-the-art edge detector:

  Edge point = strong response to smoothed image gradient $\nabla(\phi * f) = (\nabla\phi) * f$



- Multiscale version in 2-D or 3-D

  Substitute $\nabla\phi$ by gradient-like Riesz wavelet $\boldsymbol{\mathcal{R}}\psi$



Example: 3-D primal wavelet sketch

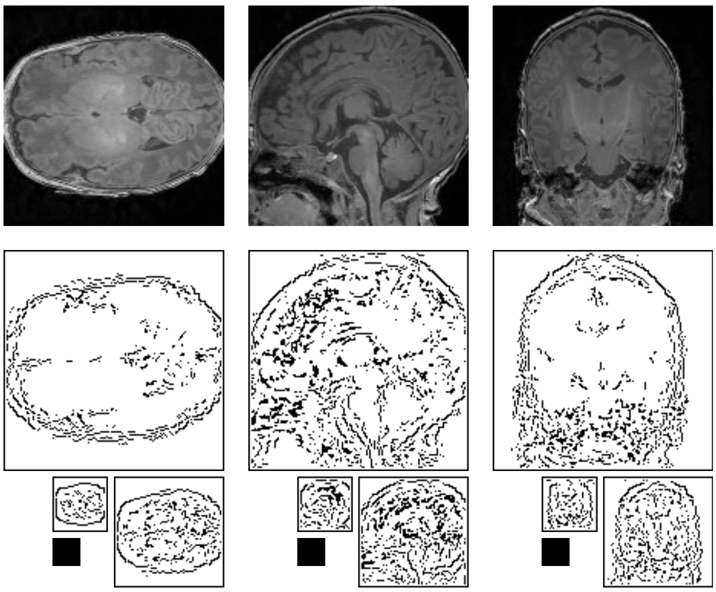
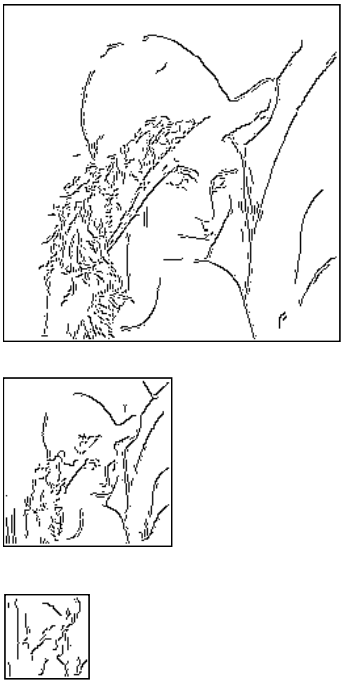


Fig. 4. Multiscale edge detection for a 3D MRI volume. Top: transverse, sagittal, and coronal slices from the original $(144 \times 144 \times 144)$ data. Bottom lines: 3D edges detected for each slice in the three analyzed wavelet bands.

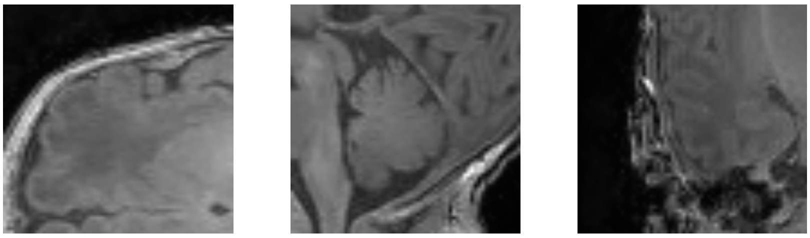
Reconstruction from primal wavelet sketch



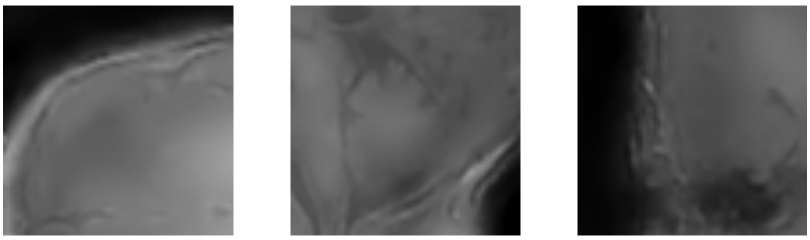
Gradient-like bandlimited Riesz wavelets

Reconstruction: $\mathbf{f} = \arg\min \left(\sum_{n \in S} \left([\mathbf{W}^T \mathbf{f}]_n - \mathbf{w}_n \right)^2 + \lambda \|\mathbf{W}^T \mathbf{f}\|_1 \right)$ as $\lambda \to 0$

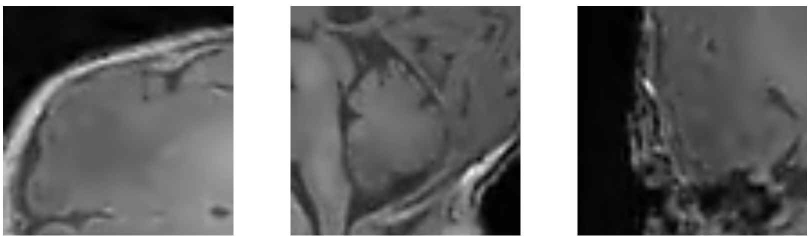
Reconstruction from 3-D wavelet sketch



(a) 2D areas from the original MRI volume.



(b) Reconstruction by orthogonal projection of the edge coefficients: 25.81 dB.



(c) Reconstructed images by solving (13) with Algorithm 1 (50 iterations): 32.06 dB.

CONTENT

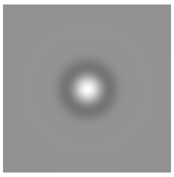
- Riesz transform and its properties ✔
- Gradient-like steerable wavelet transform ✔
- Monogenic wavelet analysis
  - Monogenic signal
  - Local wave number
  - Demodulation of holograms
- Generalizations: higher dimensions and/or higher order

Felsberg's monogenic signal analysis

- Three-component monogenic signal
  - Input signal: $f(\boldsymbol{x})$
  - Complex Riesz transform: $\underline{\mathcal{R}}f(\boldsymbol{x}) = \mathcal{R}_1 f(\boldsymbol{x}) + j\mathcal{R}_2 f(\boldsymbol{x}) = r(\boldsymbol{x})\, e^{j\theta(\boldsymbol{x})}$
  - Monogenic signal: $\mathbf{f}_{\mathrm{m}}(\boldsymbol{x}) = \big(f(\boldsymbol{x}),\, \mathcal{R}_1 f(\boldsymbol{x}),\, \mathcal{R}_2 f(\boldsymbol{x})\big) = (f,\, r\cos\theta,\, r\sin\theta)$
- Local Orientation: $\theta(\boldsymbol{x}) = \angle(\underline{\mathcal{R}}f(\boldsymbol{x}))$
- Directional Hilbert analysis: $f_\theta(\boldsymbol{x}) = f(\boldsymbol{x}) + j\mathcal{H}_\theta f(\boldsymbol{x}) = A\, e^{j\xi}$
- Local Amplitude: $A(\boldsymbol{x}) = |f_\theta(\boldsymbol{x})| = \|\mathbf{f}_{\mathrm{m}}(\boldsymbol{x})\| = \sqrt{|f(\boldsymbol{x})|^2 + |\underline{\mathcal{R}}f(\boldsymbol{x})|^2}$
- Local phase and wavenumber
  - Local phase: $\xi(\boldsymbol{x}) = \angle(f_\theta(\boldsymbol{x}))$
  - Local wavenumber: $\nu(\boldsymbol{x}) = \mathrm{D}_{\boldsymbol{\theta}}\xi(\boldsymbol{x}) = \langle \boldsymbol{\theta}, \nabla\xi(\boldsymbol{x})\rangle$ with $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$

Monogenic wavelets

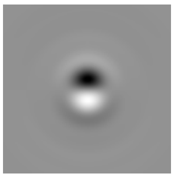
Isotropic wavelet: $\psi_{\mathrm{iso}}(\boldsymbol{x}) = (-\Delta)^{\frac{1}{2}}\phi$



$\downarrow \;\; \boldsymbol{\mathcal{R}}$ (Riesz transform)

Complex Riesz wavelet: $\underline{\mathcal{R}}\psi_{\mathrm{iso}}(\boldsymbol{x}) = \mathcal{R}_1\psi_{\mathrm{iso}}(\boldsymbol{x}) + j\mathcal{R}_2\psi_{\mathrm{iso}}(\boldsymbol{x})$

Real part



Imaginary part

Wavelet-domain monogenic analysis

- Three-component monogenic wavelet transform

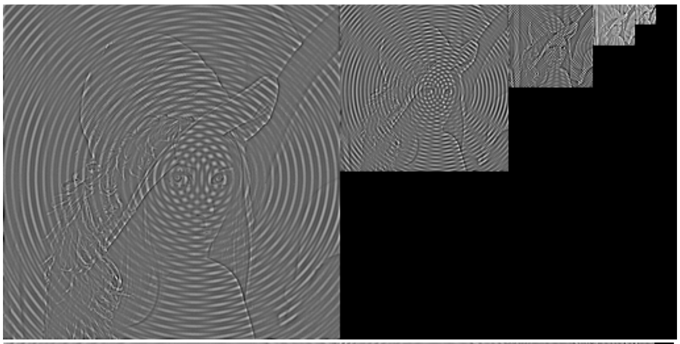
  - Real wavelet coefficients: $w_i[\boldsymbol{k}] = \langle f, \psi_{i,\boldsymbol{k}}\rangle$
  - Complex Riesz wavelet coefficients: $\underline{w}_i[\boldsymbol{k}] = \langle f, \underline{\mathcal{R}}\psi_{i,\boldsymbol{k}}\rangle = r\, e^{j\theta}$
  - Monogenic wavelet vector: $\mathbf{w}_i[\boldsymbol{k}] = \big(w_i[\boldsymbol{k}], \mathrm{Re}(\underline{w}_i[\boldsymbol{k}]), \mathrm{Im}(\underline{w}_i[\boldsymbol{k}])\big)$

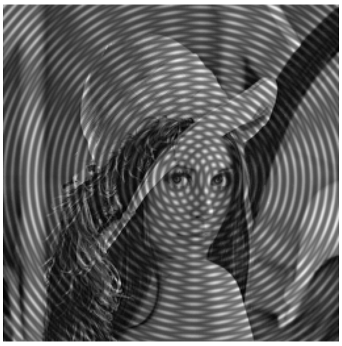
$$= (A\cos\xi, \underbrace{A\sin\xi}_{r}\cos\theta, \underbrace{A\sin\xi}_{r}\sin\theta)$$

- Local Orientation: $\theta_i(\boldsymbol{k}) = \arg(\underline{w}_i(\boldsymbol{k}))$
- Local Amplitude: $A_i(\boldsymbol{k}) = \|\mathbf{w}_i(\boldsymbol{k})\| = \sqrt{|w_i(\boldsymbol{k})|^2 + |\underline{w}_i(\boldsymbol{k})|^2}$
- Local phase: $\xi_i[\boldsymbol{k}] = \arctan\left(\frac{|\underline{w}_i(\boldsymbol{k})|}{w_i(\boldsymbol{k})}\right)$

(U.-Sage-Van De Ville, *IEEE-IP* 2009)

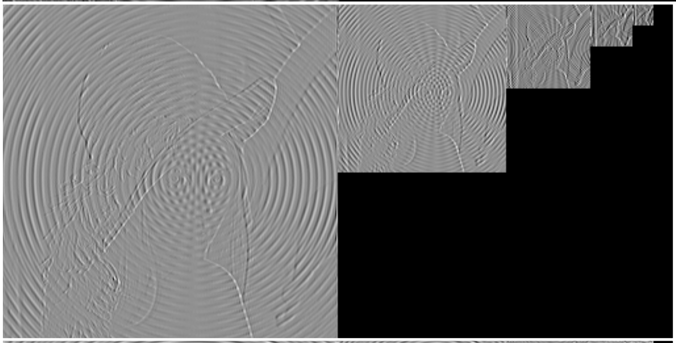
Isotropic wavelet transform



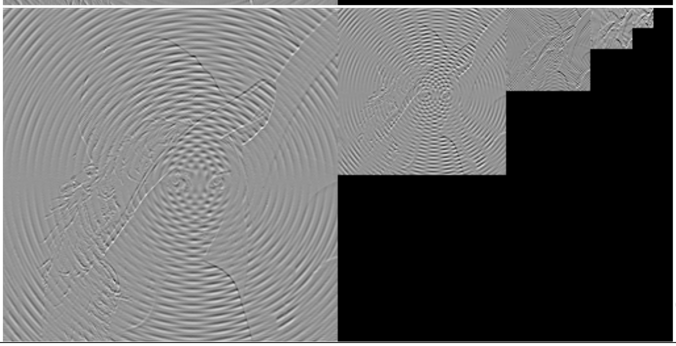


Psychedelic Lena

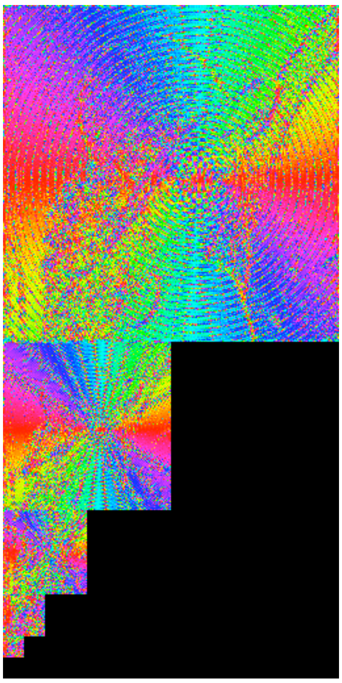
Riesz
(x-component)



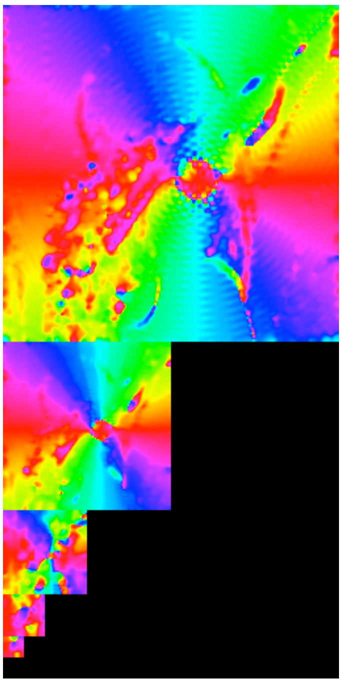
Riesz
(y-component)



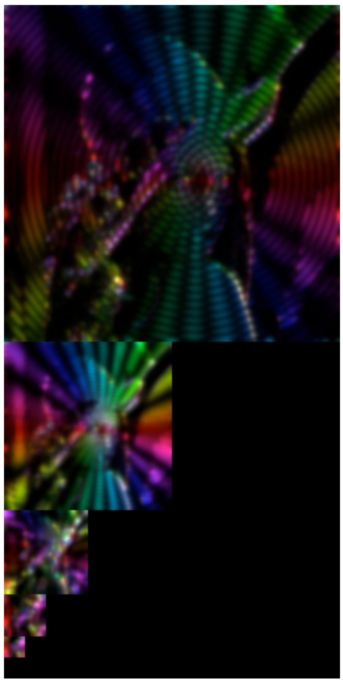
Example: Psychedelic Lena



Pointwise orientation

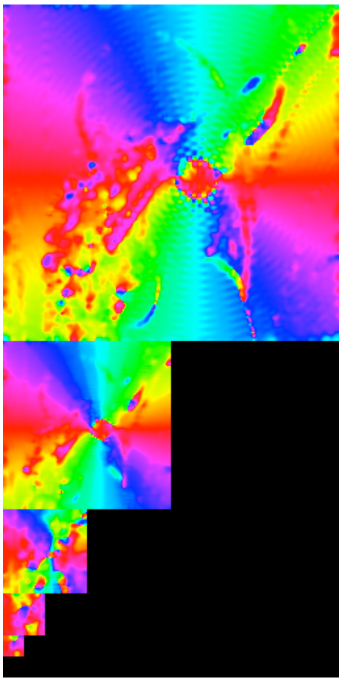


tensor orientation



Coherency in saturation
Wavelet energy in brightness

Robust tensor-based estimation: orientation & phase



- Wavelet-domain Hilbert analysis

$$w_i(\boldsymbol{x}) = (\psi_i * f)(\boldsymbol{x})$$

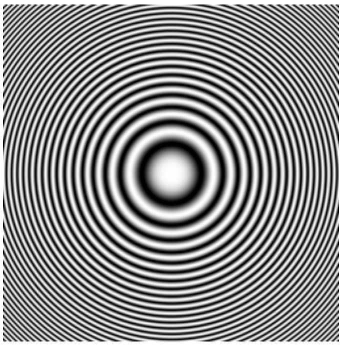
$$w_{i,\theta}(\boldsymbol{x}) = w_i(\boldsymbol{x}) + j\mathcal{H}_\theta w_i(\boldsymbol{x}) = A\, e^{j\xi}$$

Local phase: $\xi_i(\boldsymbol{x}) = \arctan\left(\frac{\mathcal{H}_\theta w_i(\boldsymbol{x})}{w_i(\boldsymbol{x})}\right)$

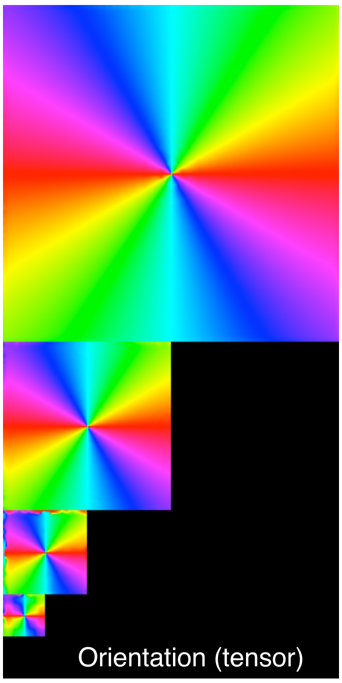
Local wavenumber: $\nu_i(\boldsymbol{x}) = \mathrm{D}_{(\cos\theta, \sin\theta)}\xi_i(\boldsymbol{x})$

Wavelet-domain orientation map

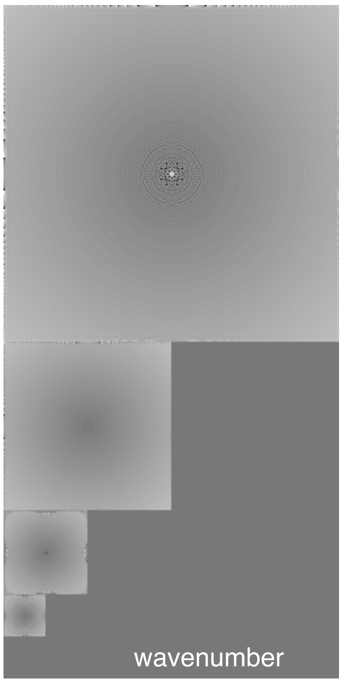
Example: Zoneplate



Synthetic zoneplate

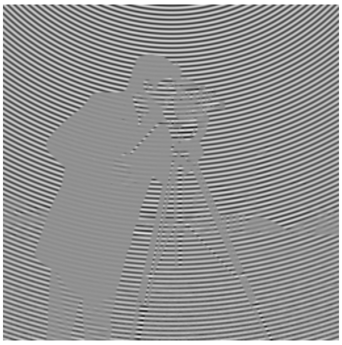


Orientation (tensor)

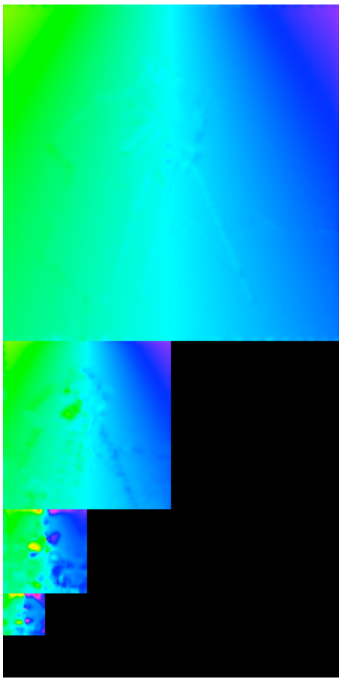


wavenumber

Example: Modulated Cameraman

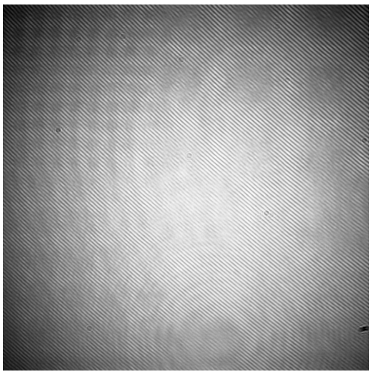


amplitude

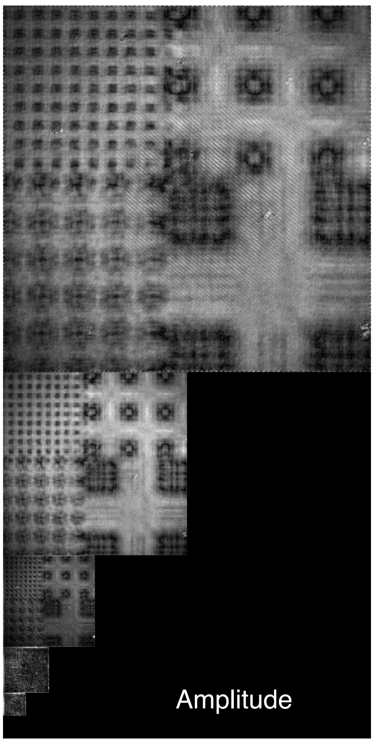


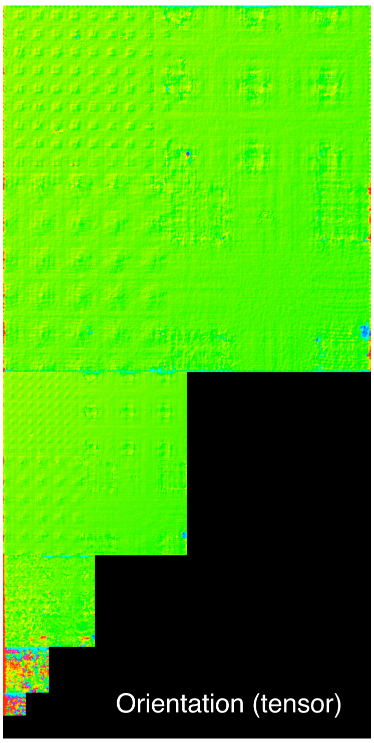
tensor orientation

Example: Digital holography microscopy



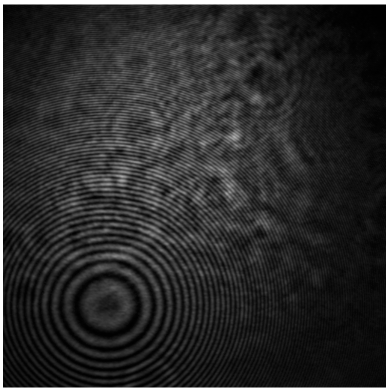
DHM image



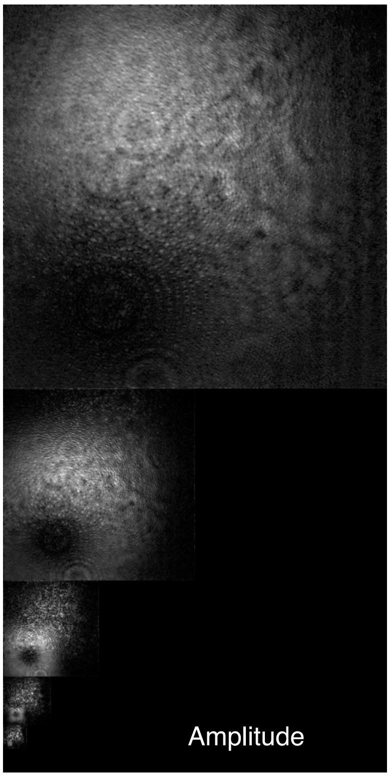


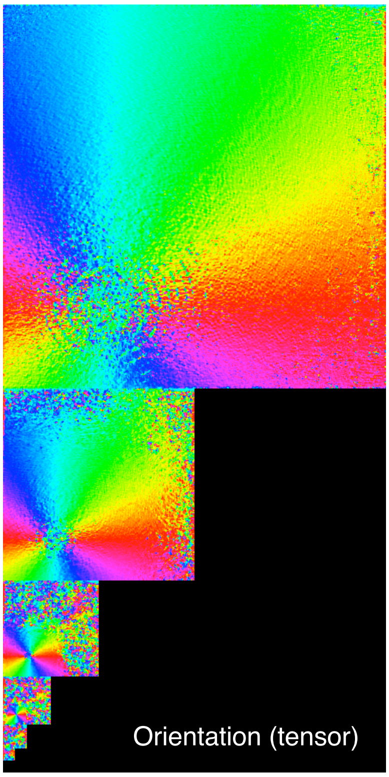
Data courtesy of Prof. Depeursinge, EPFL

Example: Digital holography microscopy



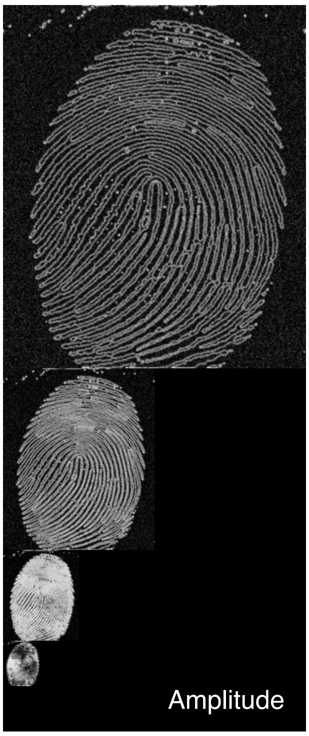
Original DHM image

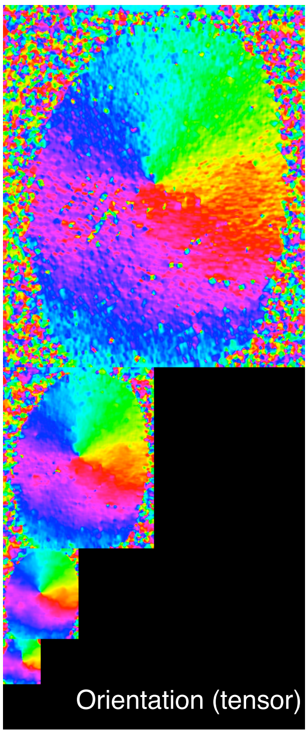




Data courtesy of Prof. Depeursinge, EPFL

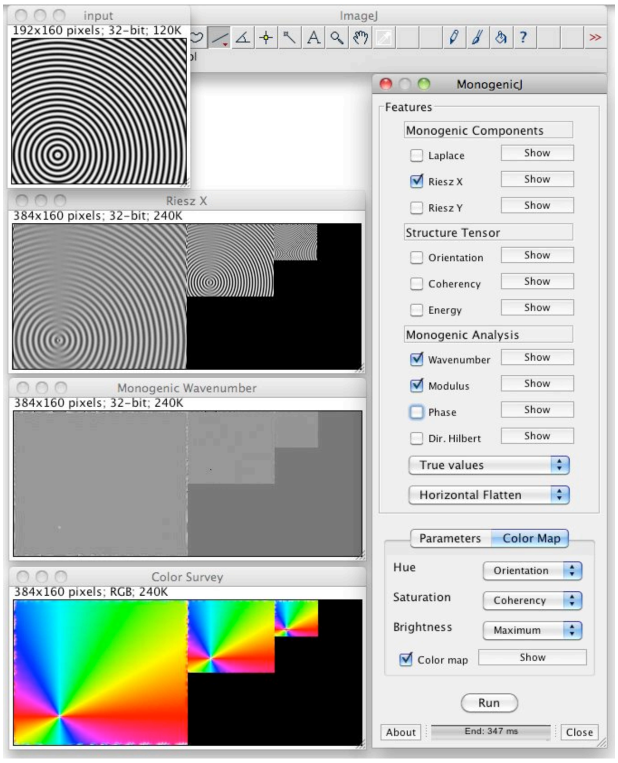
Directional wavelet analysis: Fingerprint





Wavelet-domain monogenic and structure analysis

MonogenicJ: a plugin for ImageJ (JAVA)



ImageJ
Image Processing and Analysis in Java

Author: Daniel Sage

http://bigwww.epfl.ch/demo/monogenic/

CONTENT

- Riesz transform and its properties ✔
- Gradient-like steerable wavelet transform ✔
- Monogenic wavelet analysis ✔
- Generalizations: higher dimensions and/or higher order

Higher dimensional monogenic analysis

- Directional Hilbert transform

Unit vector: $\boldsymbol{u} = (u_1, \cdots, u_d)$

$$\mathcal{H}_{\boldsymbol{u}} f(\boldsymbol{x}) = \sum_{n=1}^{d} u_n \mathcal{R}_n f(\boldsymbol{x}) = \langle \boldsymbol{u}, \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) \rangle$$

- Local Orientation: $\boldsymbol{u} = \dfrac{\boldsymbol{\mathcal{R}} f(\boldsymbol{x})}{\|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|}$

- Directional Hilbert analysis: $f_{\boldsymbol{u}}(\boldsymbol{x}) = f(\boldsymbol{x}) + j\mathcal{H}_{\boldsymbol{u}} f(\boldsymbol{x}) = A\, e^{j\xi}$

- Local Amplitude: $A(\boldsymbol{x}) = |f_{\boldsymbol{u}}(\boldsymbol{x})| = \|\mathbf{f}_{\mathrm{m}}(\boldsymbol{x})\| = \sqrt{|f(\boldsymbol{x})|^2 + \|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|^2}$

- Local phase and wavenumber
  - Local phase: $\xi(\boldsymbol{x}) = \angle(f_{\boldsymbol{u}}(\boldsymbol{x}))$
  - Local wavenumber: $\nu(\boldsymbol{x}) = \mathrm{D}_{\boldsymbol{u}} \xi(\boldsymbol{x}) = \langle \boldsymbol{u}, \nabla \xi(\boldsymbol{x}) \rangle$

Generalization: Nth-order Riesz wavelets

For all multi-indices $\mathbf{n} = (n_1, \cdots, n_d)$ such that $n_1 + \cdots + n_d = N$

- Frequency-domain wavelet formula:

$$\widehat{\psi^{\mathbf{n}}}(\boldsymbol{\omega}) = \sqrt{\frac{N!}{\mathbf{n}!}} \frac{(-j\boldsymbol{\omega})^{\mathbf{n}}}{\|\boldsymbol{\omega}\|^N} \hat{\psi}(\boldsymbol{\omega}) \quad \propto \quad (j\omega_1)^{n_1} \cdots (j\omega_d)^{n_d} \frac{\hat{\psi}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|^N}$$

- Isotropic smoothing kernel: $\phi_N(\boldsymbol{x}) = (-\Delta)^{-\frac{N}{2}} \psi(\boldsymbol{x}) = \mathcal{F}^{-1}\left\{\frac{\hat{\psi}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|^N}\right\}$

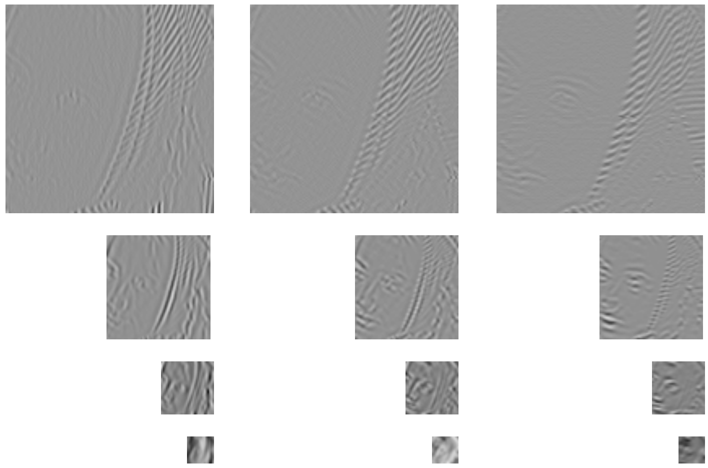
- Space-domain wavelet formula:

$$\psi^{\mathbf{n}}(\boldsymbol{x}) = \mathcal{R}^{\mathbf{n}} \psi(\boldsymbol{x}) \quad \propto \quad \frac{\partial^N}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} \phi_N(\boldsymbol{x}),$$

$$\langle f, \psi^{\mathbf{n}}(\cdot - \boldsymbol{x}) \rangle \quad \propto \quad \frac{\partial^N}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} (f * \phi_N)(\boldsymbol{x})$$

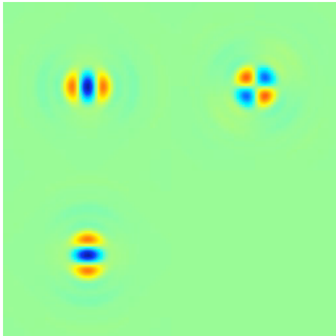
Hessian-like Riesz wavelet transform

$\mathbf{n} = (2,0)$ $\mathbf{n} = (1,1)$ $\mathbf{n} = (0,2)$



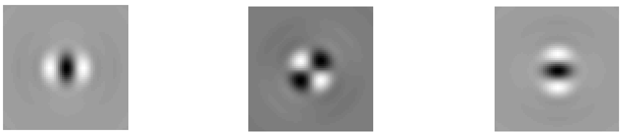
Steerability

$\cos^2\theta$ $\sqrt{2}\cos\theta\sin\theta$



$\sin^2\theta$

Second-order Riesz wavelets



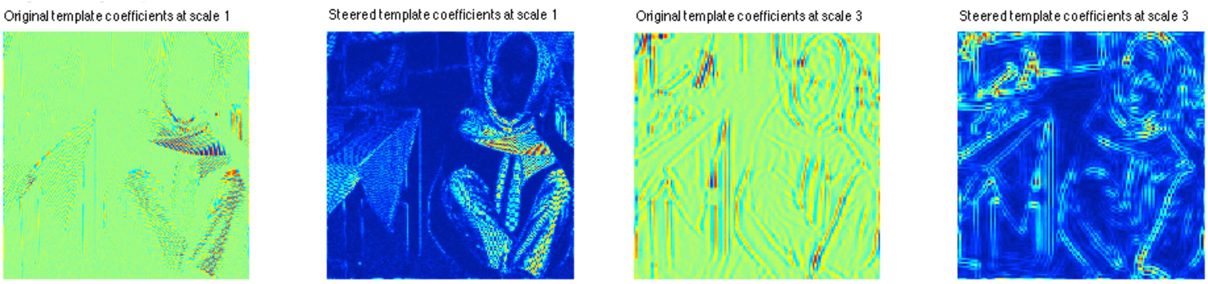
$$\left(\psi^{(2,0)} = \frac{\partial^2\phi_2}{\partial x_1^2},\quad \psi^{(1,1)} = \sqrt{2}\frac{\partial^2\phi_2}{\partial x_1 \partial x_2},\quad \psi^{(0,2)} = \frac{\partial^2\phi_2}{\partial x_2^2}\right)$$

Generalized Riesz-Wavelet Toolbox for Matlab

Toolbox Content

A toolbox that contains Matlab routines for computing the forward and backward generalized Riesz-wavelet transform of high order is provided. We have included utilities for orientation computation, coefficients steering, basic denoising, frame learning. The following demonstration routines may serve as tutorials on the use of the toolbox:

demo_Riesz2D	2D Riesz-wavelet transform decomposition and reconstruction
demo_monogenicAnalysis	Perform monogenic analysis and display estimated orientation and coherency maps
demo_monogenicAnalysis_amplitudeEqualization	Reconstruct an image with equalized monogenic amplitude.
demo_optimalTemplateSteering	Steer Riesz-wavelet coefficients to maximize the response of a template of Riesz channels
demo_basicDenoising	Perform basic denoising operations in the Riesz-wavelet domain
demo_fromRieszToSimoncelli	Demonstrate the construction of a Simoncelli's pyramid from the Riesz-wavelet frame
demo_generalizedRieszTransformLearning	Learn a generalized Riesz-wavelet frame using principal component analysis



CONCLUSION

- Riesz transform
  - Invariance properties: shift, scale, rotation, energy conservation
- Riesz-based construction of steerable wavelet transforms
  - Tight frame property (self-reversibility)
  - Multi-scale gradients
  - Rotation-invariant processing
  - Fast filterbank algorithm
- Monogenic wavelet transform/analysis
  - Local orientation, phase (shift) and wavenumber
  - Directional analysis/feature extraction
- Potential applications
  - Image reconstruction from wavelet sketch
  - Analysis/processing of fringe patterns (holography, interferometry)
  - Texture, fingerprints
  - Regularization of inverse problems

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EPFL's Biomedical Imaging Group

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SWISS NATIONAL SCIENCE FOUNDATION

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- Preprints and software: http://bigwww.epfl.ch/

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- Preprints and software: **http://bigwww.epfl.ch/**