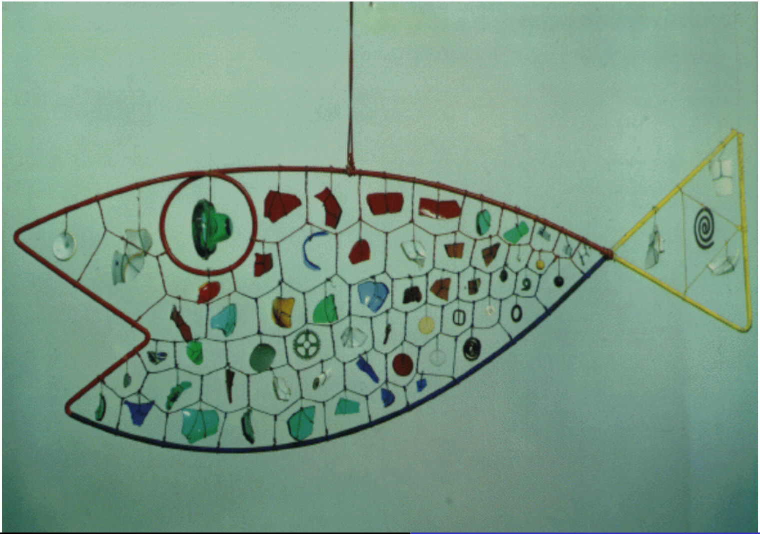
# Affine and Curved Voronoi Diagrams

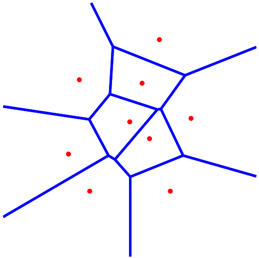
**Jean-Daniel Boissonnat**

Lectures at MPRI

# An artistic view of a Voronoi diagram



# A gallery of Voronoi diagrams



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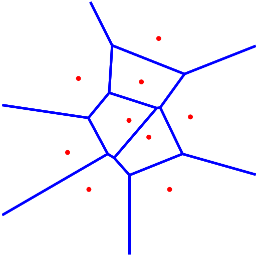

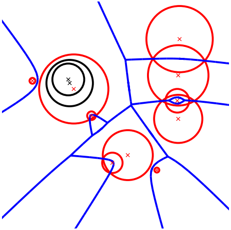

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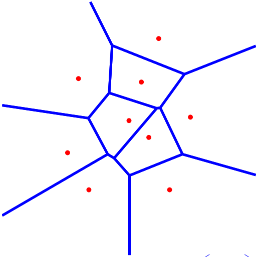

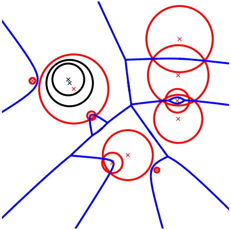

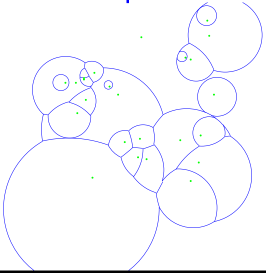

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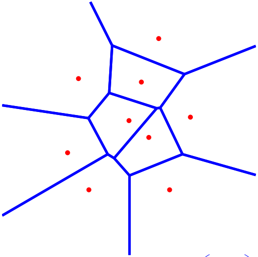
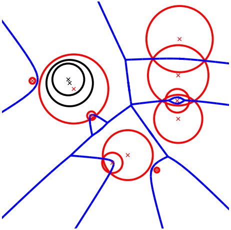
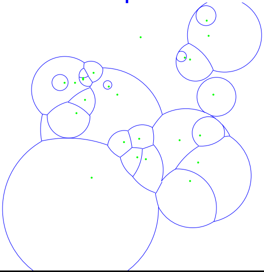
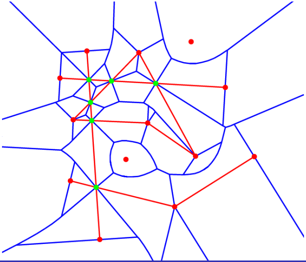




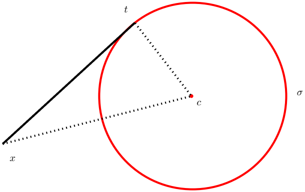




# Outline

# Power diagrams of spheres

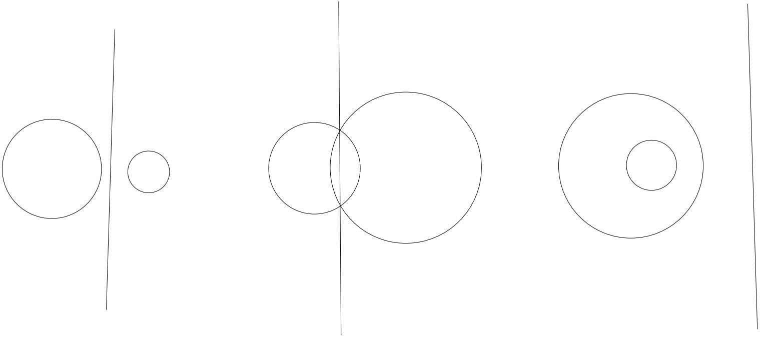
## Power of a point to a sphere



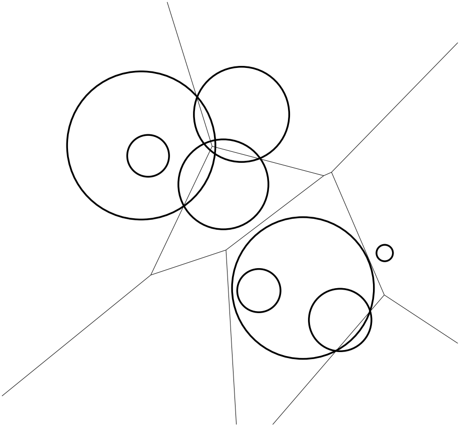

$$\sigma(x) = (x - t)^2 = (x - c)^2 - r^2$$
$$\sigma(x) < 0 \iff x \in \mathrm{int}(\sigma)$$

# Bisector of two sites = hyperplane

$$\sigma_i(x) = \sigma_j(x) \iff \not{x}^2 - 2c_i \cdot x + s_i = \not{x}^2 - 2c_j \cdot x + s_j$$



# Power diagram



Sites : $n$ spheres $\sigma_1, \ldots, \sigma_n$

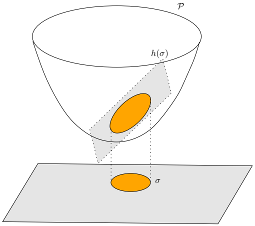
Distance of a point $x$ to $\sigma_i$

$$\sigma_i(x) = (x - c_i)^2 - r_i^2$$

$$\mathrm{Pow}(\sigma_i) = \{x : \sigma_i(x) \leq \sigma_j(x),\ \forall j\}$$

$\mathrm{Pow}(\sigma_i)$ may be empty

# Space of spheres




$\sigma \longrightarrow$ the polar hyperplane $h_\sigma$ of $\mathbb{R}^{d+1}$ : $x_{d+1} = 2c \cdot x - s$

1. If $\sigma_i = p_i$, $h_{\sigma_i}$ is the hyperplane $h_{p_i}$ tangent to the paraboloid $\mathcal{P}$
2. The vertical projection of $h_{\sigma_i} \cap \mathcal{P}$ onto $x_{d+1} = 0$ is $\sigma_i$
3. $\sigma_i(x) < \sigma_j(x) \Longleftrightarrow 2c_i \cdot x - s_i > 2c_j \cdot x - s_j$
   $\Longleftrightarrow$ at point $x$, $h_{\sigma_i}$ is above $h_{\sigma_j}$

# Space of spheres

the faces of the power diagram are the vertical projections of the faces of $\mathcal{P}(\mathcal{S}) = \bigcap_i h_{\sigma_i}^+$

The vertical projection of the dual complex $\mathcal{R}(\mathcal{S})$ of $\mathcal{P}(\mathcal{S})$ is called the regular triangulation of $\mathcal{S}$

$$\begin{array}{ccc} \mathcal{P}(\mathcal{S}) = h_{\sigma_1}^+ \cap \ldots \cap h_{\sigma_n}^+ & \longleftrightarrow & \mathcal{R}(\mathcal{S}) = \text{conv}^-(\{\phi(\sigma_1), \ldots, \phi(\sigma_n)\}) \\ \updownarrow & & \updownarrow \\ \text{power diagram of } \mathcal{S} & \longleftrightarrow & \text{Regular triangulation of } \mathcal{S} \end{array}$$

# Complexity and algorithm

nb of faces $= \Theta\left(n^{\lfloor \frac{d+1}{2} \rfloor}\right)$ (Upper Bound Th.)

can be computed in time $\Theta\left(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor}\right)$

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Main predicate

$$\texttt{power\_test}(\sigma_0, \ldots, \sigma_{d+1}) = \operatorname{sign} \begin{vmatrix} 1 & \ldots & 1 \\ c_0 & \ldots & c_{d+1} \\ c_0^2 - r_0^2 & \ldots & c_{d+1}^2 - r_{d+1}^2 \end{vmatrix}$$

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## Definition

Diagrams defined for objects and a distance function s.t. bisectors are hyperplanes

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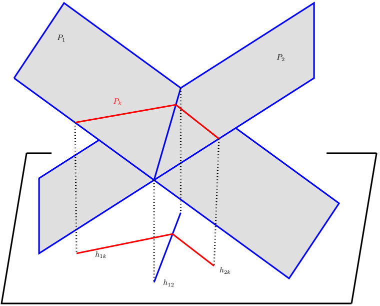
## Theorem [Aurenhammer]

*Any affine Voronoi diagram of $\mathbb{R}^d$ is the power diagram of a set of spheres of $\mathbb{R}^d$.*




$P_1$ : any non vertical hyperplane of $\mathbb{R}^{d+1}$
$P_2$ : any non vertical hyperplane such that $\text{proj}(P_1 \cap P_2) = h_{12}$

for $k \geq 3$

$P_k$ : the hyperplane such that
$\text{proj}(P_1 \cap P_k) = h_{1k}$
$\text{proj}(P_2 \cap P_k) = h_{2k}$

$$\text{proj}(P_i \cap P_j) = h_{ij} \Leftarrow \begin{array}{l} \text{proj}(P_1 \cap P_i \cap P_j) = h_{1i} \cap h_{1j} = l_{1ij} \\ \text{proj}(P_2 \cap P_i \cap P_j) = h_{2i} \cap h_{2j} = l_{2ij} \\ \text{proj}\ (\text{aff}\ (P_1 \cap P_i \cap P_j, P_2 \cap P_i \cap P_j)) = \text{aff}(l_{1ij}, l_{2ij}) = h_{ij} \end{array}$$

we define $\sigma_i = \text{proj}\ (P_i \bigcap \mathcal{P}) \quad \Rightarrow \quad h_{\sigma_i} = P_i$
$h_{ij}$ = radical hyperplane of $\sigma_i$ et $\sigma_j$

# Examples of affine diagrams

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$$\|x - a\|_Q = (x - a)^t Q(x - a) \qquad Q = Q^t$$

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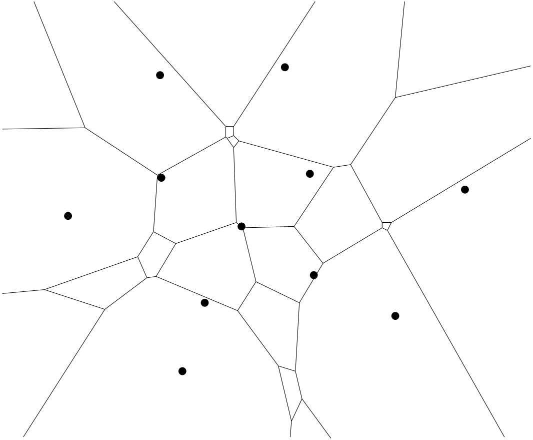
4. $k$-order Voronoi diagrams

# Order $k$ Voronoi Diagrams



Order 2 Voronoi Diagram

# A $k$-order Voronoi diagram is a power diagram

Let $E_1, E_2, \ldots$ denote the subsets of $k$ points of $E$

$$\sigma_i(x) = \frac{1}{k} \sum_{j \in E_i} (x - p_j)^2 = x^2 - \frac{2}{k} \sum_{j \in E_i} p_j \cdot x + \frac{1}{k} \sum_{j \in E_i} p_j^2$$

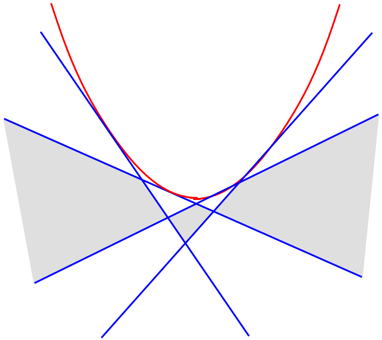
The $k$ nearest neighbors of $x$ are the points of $E_i$ iff

$$\forall j, \quad \sigma_i(x) \leq \sigma_j(x)$$

$\sigma_i$ is the sphere centered at $\frac{1}{k} \sum_{j=1}^{k} p_{i_j}$

$$\sigma_k(0) = \frac{1}{k} \sum_{j=1}^{k} p_{i_j}^2$$

# In the space of spheres



The cells of the $k$-Voronoi diagram are the projections of the cells of the $k$-th level in the arrangement of the polar hyperplanes $h_{p_i}$

# Number of faces of levels $\leq k$ in an arrangement of hyperplanes

$H$ set of $n$ hyperplanes of $\mathbb{R}^d$, $\mathcal{A}$ the associated arrangement
It is sufficient to count the number of vertices of level $\leq k$

Objects : hyperplanes of $H$

Configurations : $d$-uplets of hyperplanes ($\equiv$ a vertex of $\mathcal{A}$

Conflict : $h \in H$, $s$ vertex of $\mathcal{A}$, $s \in h^-$

The number of vertices of level $\leq k$ is equal to $|\mathcal{C}^d_{\leq k}(H)|$

# Random sampling theorem [Clarkson & Shor]

If $\mathcal{S}$ is a set of $n$ objects
$k$ an integer, $2 \leq k \leq n/(d+1)$
$\mathcal{R}_{\lfloor n/k \rfloor}$ a random subset of $\mathcal{S}$ of size $\lfloor n/k \rfloor$

$$|\mathcal{C}^d_{\leq k}(\mathcal{S})| \leq 4\,(d+1)^d\; k^d\; E(|\mathcal{C}^d_0(\mathcal{R}_{\lfloor n/k \rfloor})|)$$

**Proof :** $E(|\mathcal{C}_0^d(\mathcal{R}_r)|) = \sum_{C \in \mathcal{C}^d(\mathcal{S})} \text{Proba}(C \in \mathcal{C}_0(\mathcal{R}))$

$$= \sum_j |\mathcal{C}_j^d(\mathcal{S})| \frac{\binom{n-d-j}{r-d}}{\binom{n}{r}}$$

$$\geq |\mathcal{C}_{\leq k}^d(\mathcal{S})| \frac{\binom{n-d-k}{r-d}}{\binom{n}{r}}$$

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for $2 \leq k \leq \frac{n}{d+1}$ and $r = \lfloor n/k \rfloor$ : $\frac{\binom{n-d-k}{r-d}}{\binom{n}{r}} \geq \frac{1}{4(d+1)^d k^d}$

- By the random sampling theorem

$$|\mathcal{C}_{\leq k}(H)| = O\left(k^d\, E\left(|\mathcal{C}_0(\mathcal{R}_{\lfloor n/k \rfloor})|\right)\right)$$

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$$|\mathcal{C}_{\leq k}(H)| = O\left(k^d E\left(|\mathcal{C}_0(\mathcal{R}_{\lfloor n/k \rfloor})|\right)\right)$$

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$$|\mathcal{C}_0(\mathcal{R}_{\lfloor n/k \rfloor})| = O\left(\lfloor n/k \rfloor^{\lfloor \frac{d}{2} \rfloor}\right)$$

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- The number of vertices of level $\leq k$ is

$$O\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right)$$

# Bounds on $\leq k$-levels, $\leq k$-sets and $\leq k$-order VD

## Theorem

The total number of faces (of all dimensions) of the $k$ first levels of $\mathcal{A}$ is

$$O\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right)$$

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## Corollary

The number of vertices and faces of the $k$ first Voronoi diagrams is

$$O\left(k^{\lceil \frac{d+1}{2} \rceil} n^{\lfloor \frac{d+1}{2} \rfloor}\right)$$

# Möbius Diagrams

- Weighted points : $W_i = (p_i, \lambda_i, \mu_i)$, $p_i \in \mathbb{R}^d$, $\lambda_i \in \mathbb{R} \setminus \{0\}$, $\mu_i \in \mathbb{R}$
- Distance function :

$$\delta_M(x, W_i) = \lambda_i \|x - p_i\|^2 - \mu_i$$

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## Generalization of

- Voronoï diagrams ($\lambda_i = \lambda > 0$ et $\mu_i = 0$)
- Power diagrams ($\lambda_i = \lambda > 0$)
- multiplicatively weighted Voronoi diagrams ($\mu_i = 0$)

Bisectors are *hyperspheres*, hyperplanes or $\emptyset$

$$\lambda_i(x - p_i)^2 - \mu_i = \lambda_j(x - p_j)^2 - \mu_j$$
$$\iff \quad (\lambda_i - \lambda_j)x^2 - 2(\lambda_i p_i - \lambda_j p_j) \cdot x + \lambda_i p_i^2 - \mu_i - \lambda_j p_j^2 + \mu_j = 0$$

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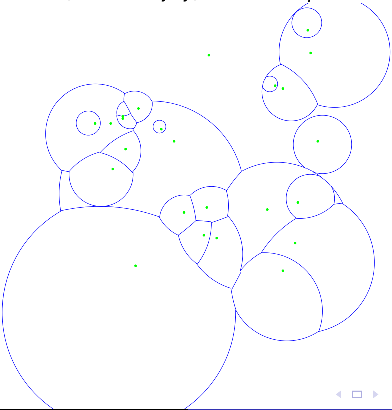

# Linearization Lemma

We can associate to each weighted point $W_i$
a hypersphere $\Sigma_i$ of $\mathbb{R}^{d+1}$ so that

the faces of the Möbius diagram of the $W_i$ are obtained by projecting vertically the faces of the restriction of the Power Diagram of the $\Sigma_i$ to the paraboloid $\mathcal{P} : x_{d+1} = x^2$

# Proof

$$\lambda_i(x-p_i)^2-\mu_i \le \lambda_j(x-p_j)^2-\mu_j$$

$$\iff (x-\lambda_i p_i)^2+(x^2+\frac{\lambda_i}{2})^2-\lambda_i^2 p_i^2-\frac{\lambda_i^2}{4}+\lambda_i p_i^2-\mu_i$$
$$\le (x-\lambda_j p_j)^2+(x^2+\frac{\lambda_j}{2})^2-\lambda_j^2 p_j^2-\frac{\lambda_j^2}{4}+\lambda_j p_j^2-\mu_j$$

$$\iff (X-C_i)^2-\rho_i^2 \le (X-C_j)^2-\rho_j^2$$

where $X=(x,x^2)\in\mathbb{R}^{d+1}$,

$C_i=(\lambda_i p_i,-\frac{\lambda_i}{2})\in\mathbb{R}^{d+1}$ and $\rho_i^2=\lambda_i^2 p_i^2+\frac{\lambda_i^2}{4}-\lambda_i p_i^2+\mu_i$

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2. *The intersection of a spherical diagram with an affine subspace is a a spherical diagram*
3. *Using stereographic projection, one can define spherical diagrams on* $S^d$
4. *The class of Möbius diagrams is identical to the class of spherical diagrams, i.e.diagrams whose bisectors are hyperspheres*

# Constructing Möbius diagrams

*The complexity of the Möbius diagram of* $n$ *doubly weighted points in* $\mathbb{R}^d$ *is* $\Theta(n^{\lfloor \frac{d}{2} \rfloor + 1})$

It can be constructed in time $\Theta(n \log n + n^{\lfloor \frac{d}{2} \rfloor + 1})$

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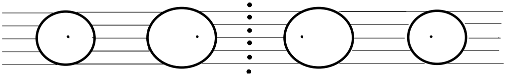

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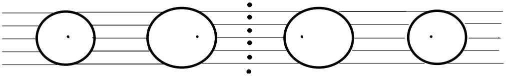


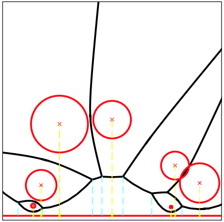
Predicates :

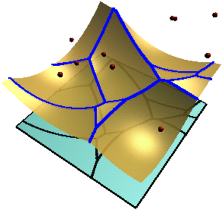
power_test

decide whether a face of Power$(\{\Sigma_i\}_{i=1}^{n})$ intersects $\mathcal{P}$

# An Euclidean model

$\sigma_0$ a hyperplane of $\mathbb{R}^d$ ($x_d = 0$)
a finite set of hyperspheres $\{\sigma_i = (p_i, \omega_i)\}_{i=1}^n$
$V(\sigma_0) = \{x \in \mathbb{R}^d : d(x, \sigma_0) \leq d(x, \sigma_i), \forall i\}$





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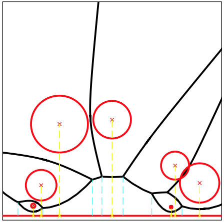
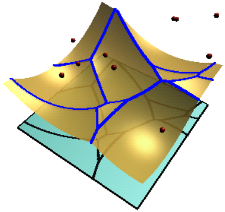
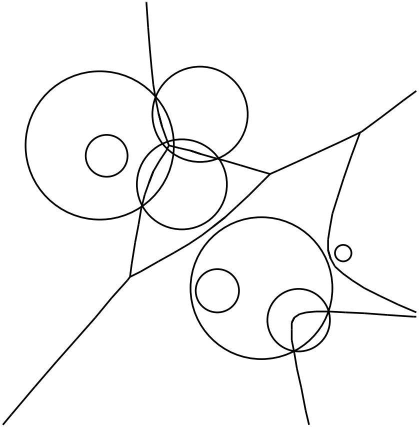




## Projection Lemma

*The vertical projection of $\partial V(\sigma_0)$ on $\sigma_0$ is a Möbius diagram*

# Apollonius diagrams of spheres



$$\sigma_i = (p_i, r_i)$$
$$\delta(x, \sigma_i) = \|x - p_i\| - r_i$$
$$\mathrm{Apo}(\sigma_i) = \{x, \delta(x, \sigma_i) \leq \delta(x, \sigma_j)\}$$

## The Projection Lemma extends to any set of spheres

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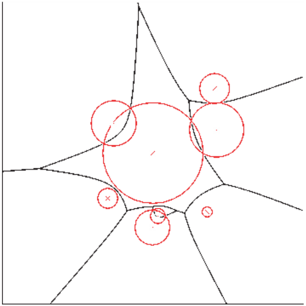

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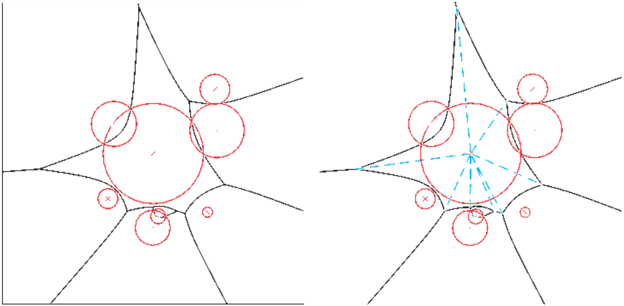

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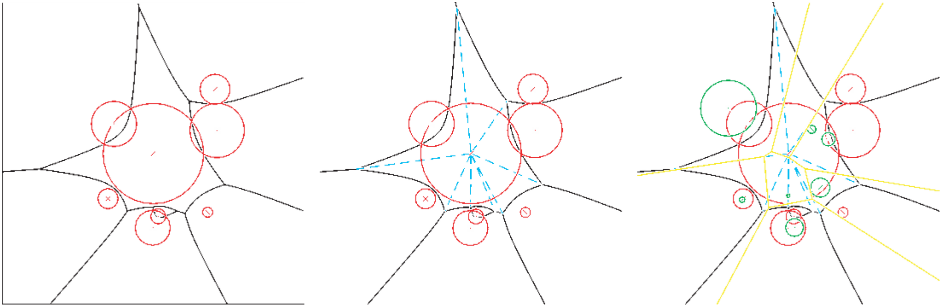

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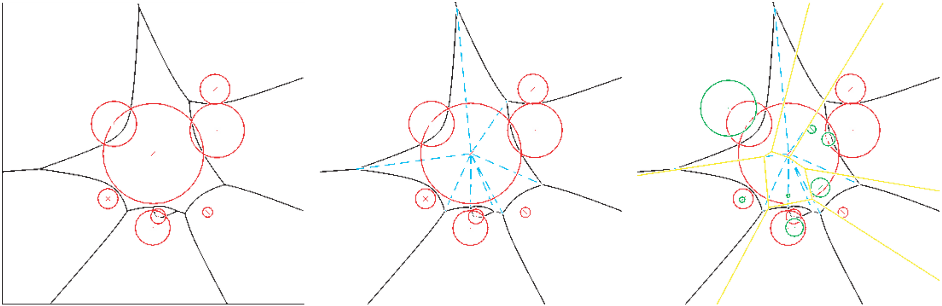

Theorem: *The combinatorial complexity of a single cell in the Apollonius diagram of n spheres of* $\mathbb{R}^d$ *is* $\Theta(n^{\lfloor \frac{d+1}{2} \rfloor})$

# CGAL implementations

## CGAL planar Apollonius diagrams [M. Karavelas]

100k circles : 40s (Pentium III, 1 GHz)

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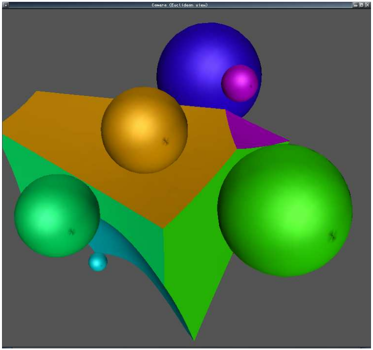
## A prototype implementation [C. Delage]



# Anisotropic Voronoi diagrams

Labelle & Shewchuk

Weighted point : $(p_i, M_i, r_i)$ where $p_i \in \mathbb{R}^d$, $M_i$ is a $d \times d$ symmetric positive definite matrix and $r_i \in \mathbb{R}$

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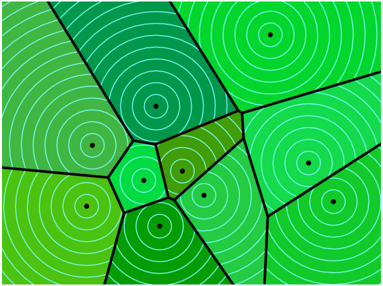
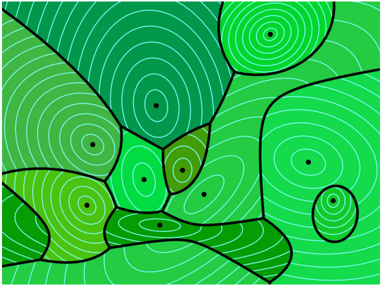


Standard diagram



Anisotropic diagram

## Linearization Lemma

In $\mathbb{R}^{\frac{d(d+3)}{2}}$, one can define a set $\Sigma$ of $n$ hyperspheres so that the anisotropic Voronoi diagram of the $n$ given weighted sites is the projection of the restriction of Pow($\Sigma$) to a $d$-manifold

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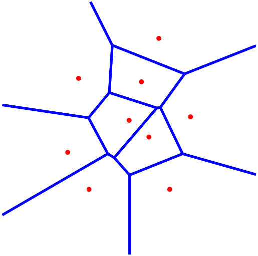
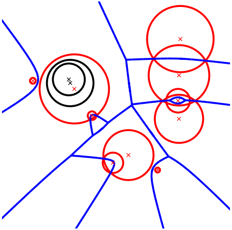
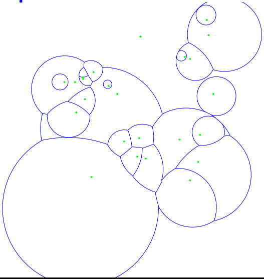
## Universality Lemma

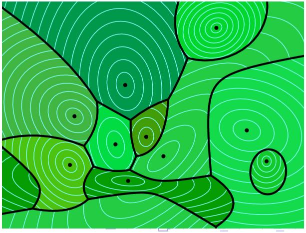
Any quadratic Voronoi diagram (i.e. with quadratic bisectors) is an anisotropic diagram

# Conclusion









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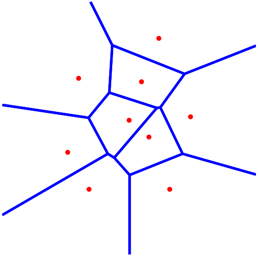
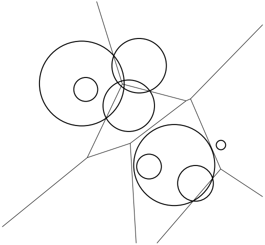
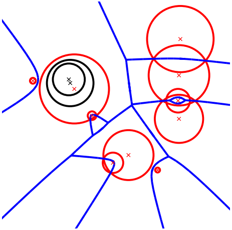
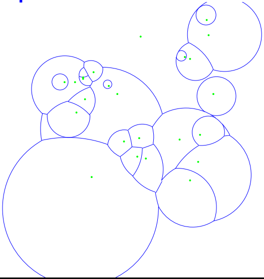
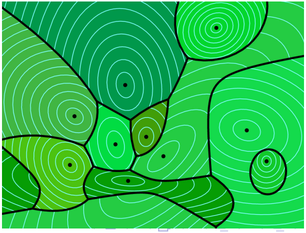

# The linearization approach

- Provides a framework for many Voronoi diagrams
- Leads to rather simple data structures and algorithms
- Robust and efficient implementations exist for simple cases

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## Further questions

- Does not directly provide good combinatorial bounds
- How to compute the restriction of an affine diagram to a manifold efficiently ?
- Approximation algorithms ?

# Acknowledgments

Menelaos Karavelas

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