- Triangulations, Delaunay triangulations
  Voronoi diagrams, the space of spheres
  Regular triangulations and power diagrams
- Constrained and constrained Delaunay triangulations
- Meshing using Delaunay refinement
- Meshing using other methods (octrees, advancing front)
- Quality of meshes
- Surface meshing
- Interpolation and reconstruction

Constrained Delaunay triangulations

Outline

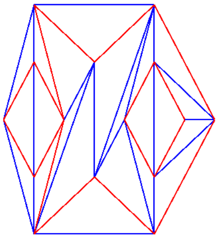
- Constrained triangulations in 2D
  existence
- Constrained Delaunay triangulations in 2D
  existence and unicity
- Optimal triangulations
  optimality of Delaunay triangulations
  when Delaunay triangulation are not optimal
- Algorithmic of constrained Delaunay triangulations
- Constrained triangulations in 3D
  existence problem
  a sufficient existence condition

Constrained triangulations in 2D

Definition

Input : a PSLG (planar straight line graph)

- a set of points P
- a set of segments S
- (P, S) is a 1dim simplicial complex i.e.
  - each endpoint of $s \in S$ is in P
  - two segments in S are disjoints or share an endpoint

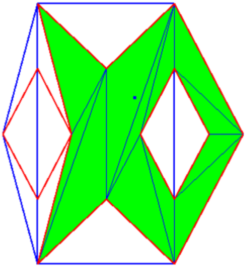


A constrained triangulation of (P, S) is a triangulation $T = T(P, S)$ such that :

- the set of vertices of T is P
- any segment $s \in S$ is an edge of T

Constrained triangulations in 2D

Application : triangulation of a polygonal region



- Build a constrained triangulation
- Mark internal facets

Constrained triangulations in 2D

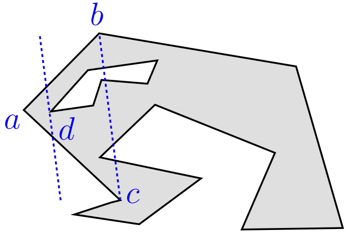
existence problem

Theorem

Any PSLG (P, S) admits a 2D constrained triangulation

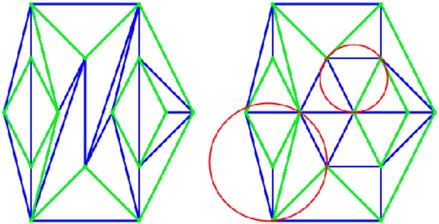
Proof.

The set of edges of any triangulation of P = a maximal set of segments with endpoints in P without intersection except at endpoints.



□

Constrained Delaunay triangulation



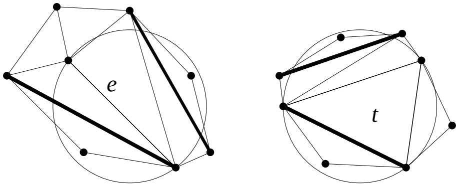
Definition 1 : Let (P, S) be a PSLG.
The constrained triangulation $T(P, S)$ is constrained Delaunay iff the circumcircle of any triangle t of T encloses no vertex visible from a point in the relative interior of t.

Visibility : p visible from q iff $\mathrm{int}(pq) \cap S = \emptyset$

Definition 2 : Let (P, S) be a PSLG.
The constrained triangulation $T(P, S)$ is constrained Delaunay iff any edge e of T is either a segment of S or is constrained Delaunay.

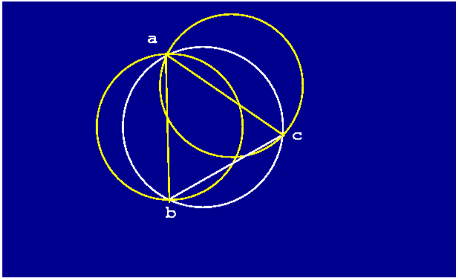
Simplex e is constrained Delaunay (cD for short) with respect to the PSLG (P, S) iff :

- $\text{int}(e) \cap S = \emptyset$
- $\exists$ a circumcircle of e that encloses no vertex visible from a point in the relative interior of e.



Constrained Delaunay triangulation

Definition 1 $\Longleftrightarrow$ Definition 2



ab, bc, ca constrained or Delaunay constrained	$\Longleftrightarrow$	circum(abc) encloses no vertex visible from $\mathrm{int}(abc)$

Constrained Delaunay triangulation

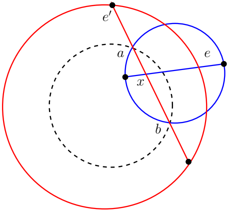
Theorem

Any PSLG (P, S) has a constrained Delaunay triangulation.
If (P, S) has no degeneracy , this triangulation is unique.

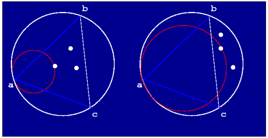
Proof.

$S \cup$ set of cD-segment $=$ a maximal set of segments
with endpoints in P
without intersection except at endpoints.

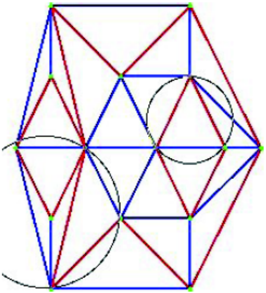
1. no intersection



2. maximal set

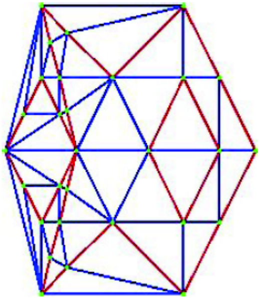


Constrained Delaunay and conform triangulations



constrained Delaunay

all simplexes are constrainded Delaunay



conform triangulation

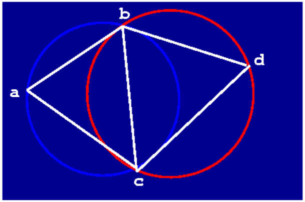
Steiner vertices on edges the triangulation is Delaunay

Locally Delaunay edges

Definition (Locally Delaunay edges (ID-edges))

Edge bc, incident to abc and abd, is locally Delaunay iff

$$a \notin \mathrm{int}(\mathrm{circumcircle}(bcd)) \iff d \notin \mathrm{int}(\mathrm{circumcircle}(abc))$$

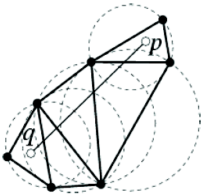


Local condition for constrained Delaunay triangulation

Theorem

Any triangulation of the PSLG (P, S) whose edges are either constrained edges or locally Delaunay edges is the constrained Delaunay triangulation of (P, S).

Proof.



p vertex visible from $q \in t$

$t_0, t_1, \dots t_n = t$ triangles intersected by pq

$\Pi(p, t_i)$ power of p wrt $\mathrm{circumcircle}(t_i)$

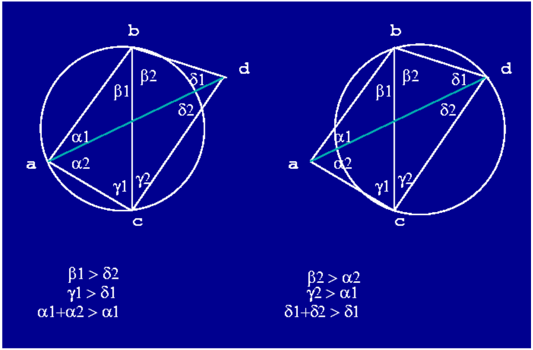
$\Pi(p, t_0) < \Pi(p, t_1) \dots < \Pi(p, t_n)$

$\Pi(p, t_0) = 0 \implies \Pi(p, t_n = t) > 0$

$\implies \mathrm{circumcircle}(t)$ encloses no vertex visible from $q \in t$

□

Delaunay flip and angular sequence



Angular sequence of a triangulation T: the sorted sequence of angles of the triangles of T

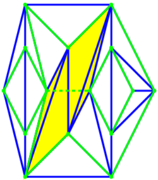
Delaunay flip : a flip that replaces a non ID-edge by an ID-edge.

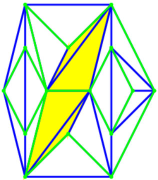
Theorem

Any Delaunay flip increases the angular sequence.

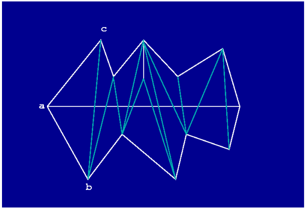
Algorithmic of 2D constrained triangulations

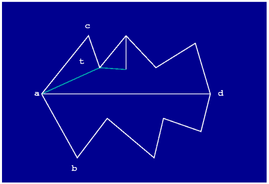
- sweep line algorithm. $O(n \log n)$
- triangulation of a polygon is $\Theta(n)$
- incremental construction
  insertion of an edge : $O(n)$ for each edge
  - insert vertices first
  - insert interior of segment next





Insertion of a constrained edge





- scan the hole boundary inserting edges into a stack
- at each step do

  while there is an ear pq, qr on top of the stack

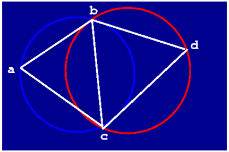
  pop pq and qr

  form triangle pqr

  push pr

A flip algorithm for constrained Delaunay triangulations

1. Start with any constrained triangulation of the PSLG (P, S)
2. Initialize a stack with edges that are neither constrained edges nor locally Delaunay edges
3. While stack is not empty
   pop edge ad from stack
   if ad is not locally Delaunay
   flip ad and update the stack,
   looking at the 4 wing edges ab, ac, db, dc



Theorem

The flip algorithm ends, and performs $O(n^2)$ flip.

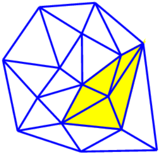
Proof 1. Use the angular sequence.

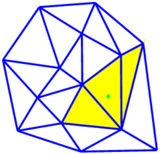
Proof 2. Use the paraboloid lift.

Algorithmic of 2D constrained Delaunay triangulations

Incremental construction

- insertion of a vertex





non Delaunay insertion of a constraint + Delaunay flips

Optimality of constrained Delaunay triangulations

Theorem

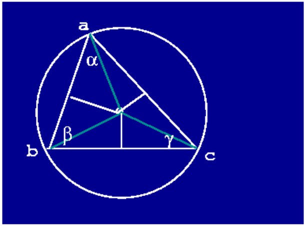
Among all the constrained triangulation of a PSLG (P, S) the constrained Delaunay triangulation optimizes:

- *the MaxMin angle*
- *the MinMax circumradius*
- *the MinMax smallest enclosing circle radius*

Proof.

The constrained Delaunay triangulation is optimal for any measure improved by a Delaunay flip □

Circumradius, angles and edge lengths



circumradius r

$$r = \frac{l_a}{2sin\alpha} = \frac{l_b}{2sin\beta} = \frac{l_c}{2sin\gamma}$$

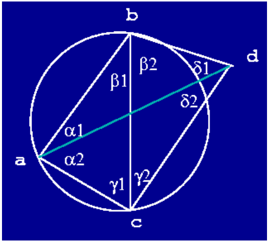
MinMax circumradius

Theorem

Delaunay flip decreases the maximum circumradius

Proof.

$$\begin{aligned} \gamma_1 &> \delta_1 \\ \beta_2 &> \alpha_2 \end{aligned}$$

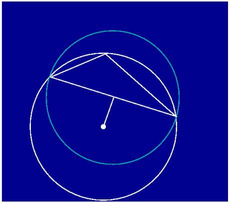


$$\text{circumradius}(abc) = \frac{ab}{2sin\gamma_1} < \text{circumradius}(abd) = \frac{ab}{2sin\delta_1}$$

$$\text{circumradius}(bcd) = \frac{cd}{2sin\beta_2} < \text{circumradius}(adc) = \frac{cd}{2sin\alpha_2}$$

□

Smallest enclosing sphere



Theorem

x_c the circumcenter of the simplex t,
x_{min} the center of the smallest enclosing sphere of t
is such that :

1. *if $x_c \in t$, $x_{min} = x_c$*
2. *otherwise x_{min} = the point of t closest to x_c*

Proof.

case 1. x_c is a minimum of the distance to farthest vertex

case 2. Let $q \in t$ closest to x_c
and f the face of t such that $q \in \mathrm{int}(f)$.
The vertices of f are equidistant to q.
smallest enclosing sphere of t = smallest enclosing sphere of f.

□

The constrained Delaunay triangulation achieves MinMax smallest enclosing sphere

Theorem

A Delaunay flip decreases the maximum smallest enclosing circle radius.

Proof.

Theorem

In any dimension, the Delaunay triangulation of a set of points minimizes the maximum smallest enclosing sphere radius.

□

The Delaunay triangulation achieves MinMax smallest enclosing radius

Proof

$t = (p_0, p_1, \dots p_d)$ a d-simplex

Barycentric coordinates

$\forall x \in R^d, \quad \lambda_i(x), i = 0 \dots p$ such that

$$x = \sum_i \lambda_i(x) p_i, \ \ \sum_i \lambda_i(x) = 1$$

Definition

$$F(t,x) = \sum_i \lambda_i(x)(p_i - x)^2 = \sum_i \lambda_i(x) p_i^2 - x^2$$

The Delaunay triangulation achieves MinMax smallest enclosing radius

(x_c, r_c) circumsphere of the simplex t.
(x_{min}, r_{min}) smallest enclosing sphere of t.

$$
\begin{aligned}
F(t,x) &= \sum_i \lambda_i(x)(p_i - x)^2 \\
F(t,x) &= \sum_i \lambda_i(x)\left((p_i - x_c)^2 + 2(p_i - x_c)(x_c - x) + (x_c - x)^2\right) \\
F(t,x) &= r_c^2 - (x - x_c)^2 = -\text{power of } \; x \; \text{ wrt } \; (x_c, r_c)
\end{aligned}
$$

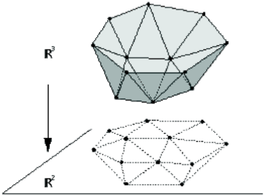
$$
\max_x F(t,x) = r_c^2 \text{ achieved for } \mathrm{x} = \mathrm{x_c} \qquad (1)
$$

$$
\max_{x \in t} F(t,x) = r_{min}^2 \text{ achieved for } \mathrm{x} = \mathrm{x_{min}} \qquad (2)
$$

The Delaunay triangulation achieves MinMax smallest enclosing radius

Lift map on the paraboloid

$$\begin{aligned} p_i &\longrightarrow \phi(p_i) = (p_i, p_i^2) \\ x &\longrightarrow \phi(x) = (x, x^2) \end{aligned}$$



$$\begin{aligned} F(t,x) &= \sum_i \lambda_i(x) p_i^2 - x^2 \\ &= \mathrm{vertical\ distance\ d}(\phi(\mathrm{t}), \phi(\mathrm{x})) \end{aligned}$$

S_P set of all simplices with vertices in P

$\min_{t \in S_P, x \in t} F(t,x)$ achieved for $t \in \mathrm{Del}(P)$

The Delaunay triangulation achieves MinMax smallest enclosing radius

P	:	a set of point
T	:	a triangulation of P
DT	:	the Delaunay triangulation of P

$$F_T(x) = F(t,x) \;\; x \in t,\;\; t \in T \qquad F_T(x_T) = \max_x F_T(t,x)$$

$$F_{DT}(x) = F(t,x) \;\; x \in t,\;\; t \in DT \qquad F_{DT}(x_{DT}) = \max_x F_{DT}(t,x)$$

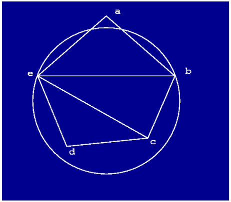
$$\max_{t \in T} r_{min}(t)^2 = F_T(x_T) \geq F_T(x_{DT}) \geq F_{DT}(x_{DT}) = \max_{t \in DT} r_{min}(t)^2$$

When Delaunay flip does not work

Delaunay triangulation does not optimize

- MinMax angle
- MaxMin elevation
- Total edge length

Using a flip to locally optimize a measure
which is not optimized by Delaunay triangulation
may leads to a lock.



example MinMax angle

$$\hat{c} > \hat{d} > \hat{e} = \hat{b} > \hat{a}$$

optimal triangulation : ad, ac
blocked situation : eb,ec

When Delaunay flip does not work

Two solutions to get out from a local minimum

- simulated anealing : allow flips which do not improve the triangulation measure
- Have more powerfull local optimization operations, e.g. edge insertion

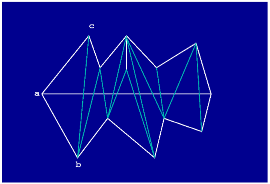
Edge insertion

Measure of the triangulation to be optimized :
$f(T) = \min_{t\in T}$ or $\max_{t\in T} f(t)$
example : $f(t) =$ max angle of t, $f(T) = \max_{t\in T} f(t)$

Anchored measure
Triangle abc has an anchor in a iff any triangulation T such that $f(T) < f(abc)$ has an edge ad intersecting bc.
A measure is anchored iff any triangle has an anchor.

Basic operation : edge insertion
insertion of edge ad means :

- remove all edges intersecting ad
- retriangulated the two regions R_1 and R_2 formed when adding edge ad



Edge insertion

Theorem

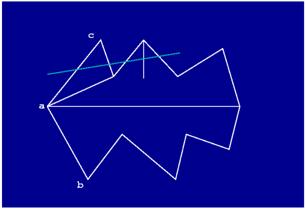
Any anchored measure can be optimized through edge insertion

Proof.

While T is not optimal, there is an edge insertion improving the measure.
Let $t = abc$ such that $f(T) = f(t)$ and a be the anchor of t.
Let ad be the edge intersecting bc in the optimal triangulation T^*
Inserting ad improves the measure.

When inserting ad, regions R_1 and R_2 can be triangulated so that :
$f(T(R_1)) < f(abc)$ and $f(T(R_2)) < f(abc)$

There is always an ear $t_1 = pqr$ of R_1 chopped by an edge of T^*.
$f(pqr) < f(abc)$:
T^* does not break anchor q
T does not break anchor p and r



Optimal triangulation through edge insertion

Algorithm

Initialize $T = T(P, S)$ a constrained triangulation
While
  there is a triangle $t = abc$ with $f(t) = f(T)$
  there is a free edge ad breaking the anchor of t
do
  insertion of ad yields triangulation T'
  if $f(T') < f(T)$, $T = T'$
  otherwise eliminate ad

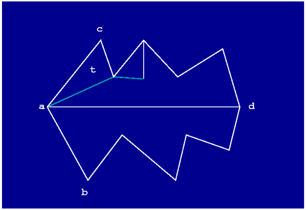
free edge = edge intersecting no constrained edge
not yet eliminated

Complexity : $O(n^3)$
total nb of edges :$O(n^2)$
complexity of an insertion $O(n)$

Optimal triangulation through edge insertion

Triangulation of regions R_1 and R_2

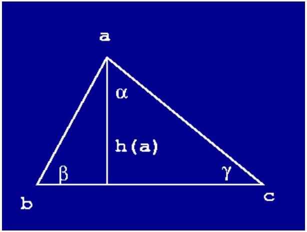
While there is an ear t
such that $f(t) < f(abc)$,
add $t = abc$ to the triangulation.
(Use a stack as in Graham walk)



- If triangulation of R_1 and R_2 ends up yielding $f(T') \leq f(abc)$ edge bc will never appear again
- Otherwise edge ad is eliminated.

MaxMin elevation

Elevations of a triangle $t = abc$
$h(a) = ab\sin\beta = ac\sin\gamma$
$h(b) = bc\sin\gamma = ba\sin\alpha$
$h(c) = ca\sin\alpha = cb\sin\beta$



$\sin\alpha \leq \sin\beta \leq \sin\gamma \Longrightarrow h(a) \geq h(b) \geq h(c)$
The smallest elevation arises from the vertex with maximum angle

Theorem

Min elevation is an anchored measure.

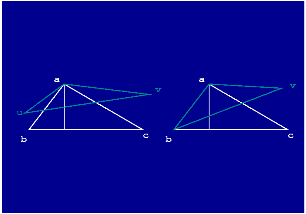
Proof.

Let $t = abc$ be a triangle with $h_{min}(t) = h(a)$

Any triangulation T such that

- $t \notin T$
- T does not break anchor of t in a

is such that $h_{min}(t) < h(a)$

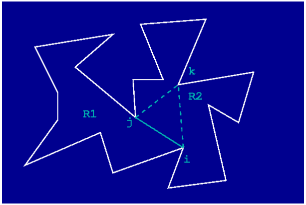


□

Optimal triangulation of a polygon through dynamic programming

Decomposable measure

- $f(T(R)) = g(f(T(R_1)), f(T(R_2)), i.j)$
- g can be computed in time O(1)
- g is monotonous wrt $f(T(R_i))$
- $f(t)$ can be computed in time $O(1)$



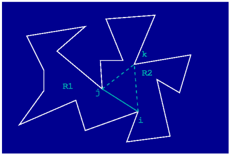
Examples of decomposable measure:

min or max angle
min elevation
total edge lentgh

Optimal triangulation of a polygon through dynamic programming

R_{ij} polygon with vertices $i, i+1 \dots j$

$$\begin{aligned} F(i,j) &= +\infty \ \text{ if } \ \mathrm{ij} \cap \partial \mathrm{R} \neq \emptyset \\ F(i,j) &= Min_T f(T(R_{ij})) \ \text{ otherwise} \\ &= \min_{i<k<j} g\left(g(F(i,k), ijk, j, k), F(k,j), k, j\right) \end{aligned}$$



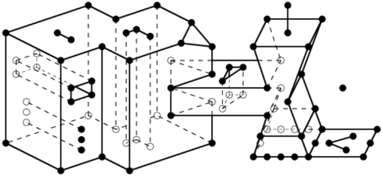
$Min_T F(T(R)) = F(1,n)$
Compute $F(i,j)$ in increasing order of j and decreasing order of i

Complexity : $O(n^3)$
can be improved to $O(En)$ or even $O(n^2 + E^{3/2})$
where $E = O(n^2)$ is the nb of edges in the visibility graph

Constrained triangulation in 3D

Input : A piecewise linear complex (PLC) C, i.e.
a set of faces of dimension 0,1,2 (vertices, edges, facets) such that :

- the boundary of any face of C is the union of faces of C
- the intersection of two faces of C is either empty
  or the union of faces of C



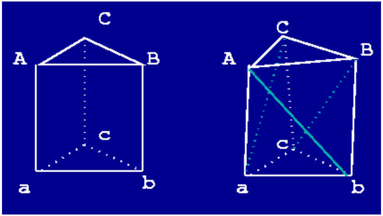
Ouptut : A 3D triangulation $T(C)$ such that :

- vertex set of C = vertex set of $T(C)$
- any edge of C is an edge of $T(C)$
- any facet of C is the union of faces of $T(C)$

Constrained triangulation in 3D

In 3D, constrained triangulations do not always exist.

Schönhardt polyedra
cannot be triangulated without adding extra (Steiner) vertices

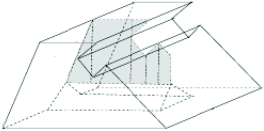


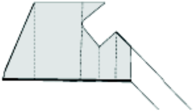
Forbidden edges
aB, bC, cA

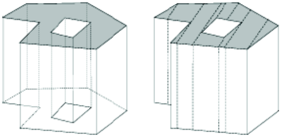
Types of tetrahedra
$ABCa$
$ABAc$
$ABab$

Vertical decomposition of a polyhedra

Complexity $O(n^2)$





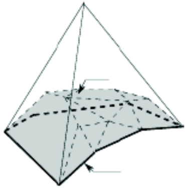


Triangulation of a polyhedra

Triangulation of a polyhedra

1. Elimination of convex vertices
2. Vertical decomposition

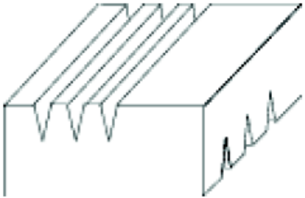
Yields a triangulation of size $O(n + r^2)$
r nb of reflex edges



Theorem

There are polyhedra with n vertices,
any triangulation of which is $\Theta(n^2)$

Proof.



n notches on the paraboloid $z = xy$
n notches on the paraboloid $z = xy + \epsilon$
Any convex included in the polyhedra has a volume $\leq 1/n^2$ □

A Delaunay edge : there is a circumsphere enclosing no vertex.
A strongly Delaunay edge: there is a circumsphere enclosing no vertex and passing through no other vertex.

Theorem

Any PLC such that :

- *the edges are stongly Delaunay*
- *there is no subset of five co-spherical vertices*

has a constrained triangulation
(which is in fact a constrained Delaunay triangulation).

Remark : "strongly" is necessary.
think of Schonhart polyhedra.

Constrained facets and constrained subfacets

Let C be a PLC whose edges are strongly Delaunay edges.
Let f be a facet of C and h_f the supporting hyperplan of f.
Let V_f be the subset of vertices of C in h_f,
and let $\mathrm{Del}(V_f)$ be the 2D Delaunay triangulation of V_f.
Being strongly Delaunay, any edge e of C included in h_f,
is an edge of $\mathrm{Del}(V_f)$.

Constrained subfacets : the triangle $t \in \mathrm{Del}(V_f)$ that are included in f.

3D constrained Delaunay triangulation

sketch of the proof of the existence condition.

Constrained Delaunay simplices. Let C be a PLC.
A simplex s with vertices in C is said to be constrained Delaunay if

- $\mathrm{int}(s)$ intersects no face f of C except if $s \subset f$.
- there is a circumsphere of s enclosing no vertex of C visible from some point in $\mathrm{int}(s)$.
  Obstacle to visibility are the (open) facets of C.

Proof of the existence condition.

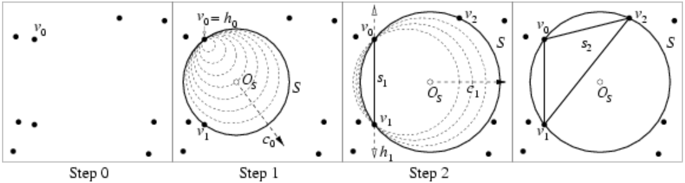
We show that the set of constrained Delaunay (cD) tetrahedra form a constrained triangulation of the PLC C.

1. Any point in $\mathrm{conv}(C)$ is included in a cD tetrahedra.
2. cD tetrahedra form a simplicial complex.
3. Any constrained subfacets is a facet of a cD tetrahedra.

□

This triangulation is called the constrained Delaunay triangulation of C

Building constrained Delaunay tetrahedra

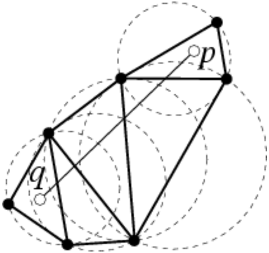


Let s be a k-cD simplex
S a circumsphere of s empty of visible vertex.
h a hyperplane including s
h^+ halfspace bounded by h
Move S in the pencil sharing $S \cap h$, growing $S \cap h+$
until S encounters a vertex u of C visible from some point of $\mathrm{int}(s)$
Growing sphere th. (below) $\Longrightarrow \mathrm{conv}(s, u)$ is a $(k+1)$-cD simplex.

1. Any $p \in \mathrm{conv}(C)$ belongs to a cD tet

For any point $p \in \mathrm{conv}(C)$, we build a cD tet including p.

1. build a first cD tet t
2. if $p \in t$, done
   else let $q \in t$
3. repeat while $p \notin t$
   $t =$ cD tet
   grown from the facet f of t
   intersected by qp.

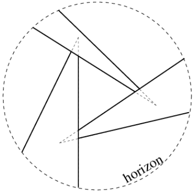


Carefull : growing a cD tet from a cD subfacet f requires the existence of a vertex of C in h_f^+ visible from some $p \in f$. (Visibility th. below)

Visibility theorem

Theorem (Visibility theorem)

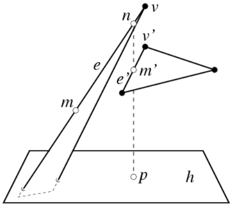
Let C be a PLC with strongly Delaunay edges,
h be a hyperplan and p a point of h.
If there is a vertex of C in the halfspace h^+,
there is a vertex of C in h^+, visible from p



covering edges

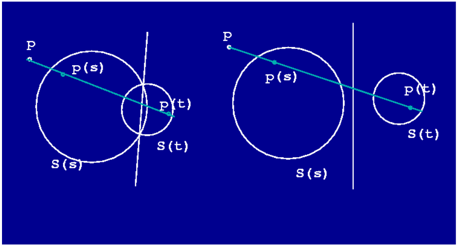
s, t edges of C, s covers t from p if :
$\exists p_s \in s \text{ and } p_t \in t \text{ with } p_s \in pp_t$

if there is no vertex in h^+,
visible from p,
there is a cycle of covering edges.



End of the visibility theorem proof.

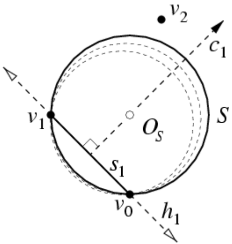
s and t strongly Delaunay edges
with empty circumspheres $S(s)$ and $S(t)$.
If s covers t from p,
$\mathrm{power}(p, S(s)) > \mathrm{power}(p, S(t))$,
$\Longrightarrow$ there is no cycle of covering edges.



Growing sphere theorem

Theorem (Growing sphere theorem)

Let C be a PLC with strongly Delaunay edges.
Let s be a cD simplex (edge or facet).
If u is a vertex visible from $p \in \mathrm{int}(s)$ such that the circumsphere $S(s, u)$ encloses no vertex of C visible from some point of $\mathrm{int}(s)$,
$\mathrm{conv}(s, u)$ *is a cD simplex.*



Proof in two steps.

1. Any point $r \in s$ is visible from u.
2. The sphere $S(s, u)$ encloses no vertex of C visible from some point of $\mathrm{int}(\mathrm{conv}(s, u))$

□

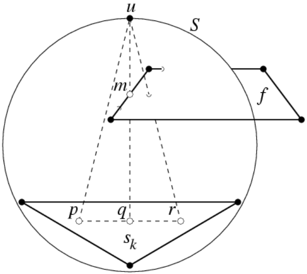
Proof of the growing sphere theorem

Step 1.

C be a PLC with strongly Delaunay edges.
s is a cD simplex (edge or facet).
u is a vertex visible from $p \in \text{int}(s)$ such that
the circumsphere $S(s, u)$ encloses no vertex of C
visible from some point of $\text{int}(s)$.

Step 1. Any point $r \in s$ is visible from u.

Proof.



Assume the reverse. Then, there are an edge $e \in C$ and a point $q \in s$ st :

- $e \cap uq = \text{a point m}$
- m visible from p.

Then,

- there is a vertex of e in $S(s, u)$ (by Lemma 1.1 below)
- there is a vertex of C in $S(s, u)$ visible from p (by Lemma 1.2 below)

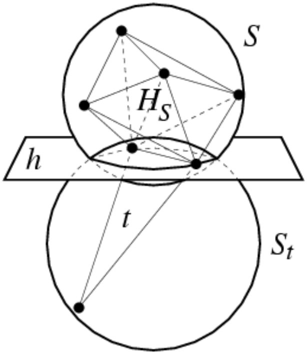
□

Lemma 1.1

for the proof of the growing sphere theorem

Lemma (Lemma 1.1)

Let S be a sphere,
H_S the convex hull of vertices of C in S
e a strongly Delaunay edge
intersecting $\mathrm{int}(H_S)$.
Then, one of the vertices of e is in S



Proof.

S_e empty circumsphere of e.
h radical hyperplan of S and S_e,
h^+ halfspace with smaller power to S than to S_e
e is strongly Delaunay $\Longrightarrow$

- any vertex in H_S is in h^+
- e has at least one vertex in h^+ hence in S

□

Lemma 1.2

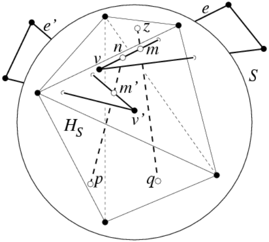
for the proof of the growing sphere theorem

Lemma (Lemma 1.2)

Let C be a PLC with strongly Delaunay edges.
Let S be a sphere and p a point in S .
If there is an edge e of C

- *with an endpoint ν in S*
- *and a point $m \in e \cup S$ visible from p,*

then there is a vertex of C in S visible from p.



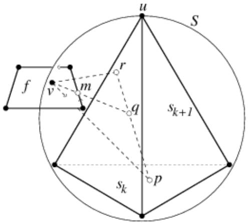
Proof.

If the endpoint of ν of e is not visible from p
we find an edge e' covering e from p.
There is a point $n \in e' \cup S$ visible from p
and e' has an endpoint in S (by lemma 1.1)
Repeating, we find
- either a vertex visible from p
- or a cycle of covering edges from p. □

Proof of the growing sphere theorem

Step 2

C be a PLC with strongly Delaunay edges.
s is a cD simplex (edge or facet).
u is a vertex visible from $p \in \mathrm{int}(s)$ such that the circumsphere $S(s, u)$ encloses no vertex visible from some point of $\mathrm{int}(s)$.
Step 2. The sphere $S(s, u)$ encloses no vertex of C visible from some point in $\mathrm{int}(\mathrm{conv}(s, u))$



Proof.

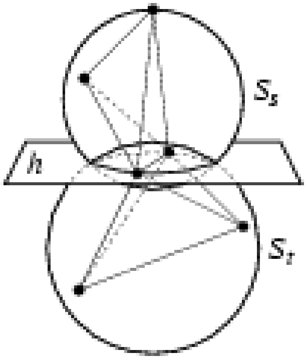
Assume for contradiction that $S(s, u)$ encloses v visible from $r \in \mathrm{int}(\mathrm{conv}(s, u))$.
We find an edge $e \in C$ and a point $m \in S \cap e$ visible from p as in Step 1.
Hence there is a vertex of C visible from p. □

2. cD simplices form a simplicial complex

Two cD tet are
- either disjoint
- or share a lower dimensional
  common face

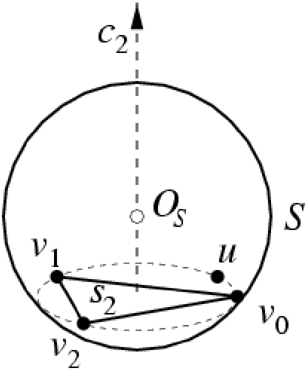
Proof.

t_1 cD tet with circumsphere S_1
t_2 cD tet with circumsphere S_2
h radical hyperplan of S_1 and S_2
vertices of t_1 are in halfspace h^+
vertices of t_2 are in halfspace h^- □



Any constrained subfacets is a facet of a cD tetrahedra.

Growing a cD tetrahedra from a constrained facet.



A 3D constrained Delaunay triangulation algorithm

Input : A PLC C
Output : A triangulation T such that :
- any vertex of C is a vertex of T
- any edge or facet in C is a union of faces in T

1. Initialize T = Delaunay triangulation of vertices of C
2. While some edge e in C is not strongly Delaunay
   split edge e
3. While some subfacet f in C is not in T
   - delete tetrahedra in T intersected by f
   - add f
   - triangulate both part of the hole.