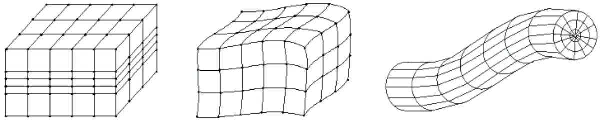
Introduction to meshing

What is a mesh?

A mesh is a cellular complex
partitioning a given object or domain
into elementary cells

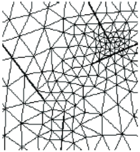
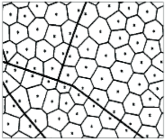
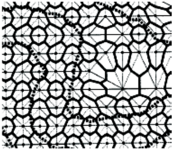
Elementary cells : cells admits a bounded description
Cellular complex : two cells are disjoint
or share a lower dimensional faces



Structured meshes

Every vertex has the same combinatorial environnement i.e the same number of incident faces of any dimension.

- ▷ economic storage, efficient for e.g. FEM applications

Unstructured meshes

Mostly simplicial meshes.

▷ highly flexible to fit the domain geometry.

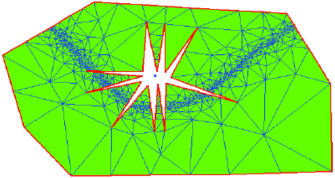
Application domains

Meshes are used in the following domains

- Graphics applications
- Scientific computing
  - solving PDE through finite elements
- Simulation
  - crack simulations, fluid dynamics etc...
- Modelisation
  - CAD-CAM applications
  - shape numerisation
  - medical imaging

The goals of a mesh generator

- respect boundaries and internal constraints
- cells according to shape criteria
- edge length according to size requirement
- control of the number of vertices



Triangulations and Meshes

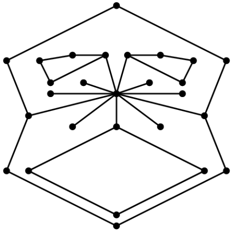
Outline

- Triangulations, Delaunay triangulations
  Voronoi dagrams, the space of spheres
  Regular triangulations and power diagrams
- Constrained and Delaunay constrained triangulations
- Meshing using Delaunay refinement
- Meshing using other methods (octrees, advancing front)
- Quality of meshes

The 2D meshing problem

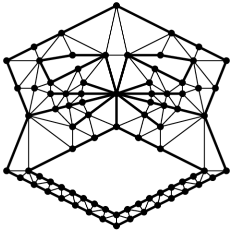
Input :

- a PSLG C
  (planar straight line graph)
- a bounded domain Ω to be meshed.
  Ω is bounded by some edges in C



Output : a mesh of domain Ω
i. e. a triangulation T such that

- vertices of C are vertices of T
- edges of C are union of edges in T
- the triangles of T that are $\subset \Omega$
  have controlled size and quality

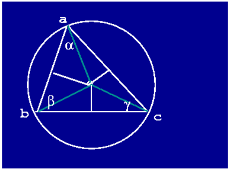


Quality measures of triangle

minimum angle α
maximum angle $2\pi - \alpha$

radius-edge ratio

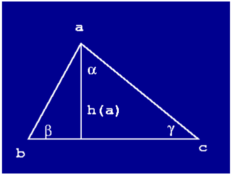
$$\rho = \frac{\text{cirmcumradius}}{\text{min edge length}} = \frac{1}{2\sin\alpha}$$



edge-elevation ratio

$$\rho_h = \frac{\text{max edge length}}{\text{min elevation length}}$$

$$\frac{1}{\sin\alpha} \leq \rho_h \leq \frac{2}{\sin\alpha}$$

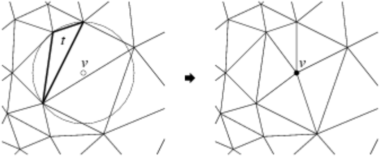


radius-radius ratio

$$\rho_i = \frac{\text{circumradius}}{\text{inscribed circle radius}}$$

$$\frac{1}{\sin\alpha} \leq \rho_i \leq \frac{3}{2\sin^2\alpha}$$

Base of Delaunay refinement 1.



- use Delaunay (and constrained Delaunay) triangulations
- kill bad triangles by circumcenter insertion

Definition (Bad triangle)

A triangle is bad if :
- either is oversized
- or its radius-edge ratio ρ is greater than a constant B.

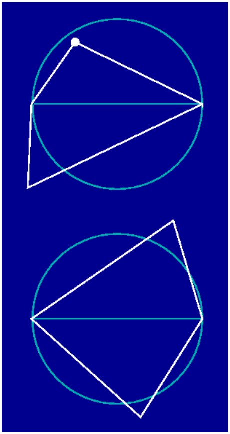
$$\rho \geq B \iff \sin\alpha \leq \frac{1}{2B}$$
$$\iff \alpha \leq \arcsin\frac{1}{2B}$$

Base of Delaunay refinement 2.

C PSLG describing the constraints
T triangulation to be refined in a mesh

Respect of the PSLG

- Edges a C are split until constrained edges are edges of T
- Constrained edges are required to be Gabriel edges.



Gabriel edges

An edge of a triangulation is a Gabriel edge
if its smallest circumcirle encloses no vertex of T

Encroachment

An edge e is encroached by point p
if the smallest circumcirle of e encloses p.

Delaunay refinement alogrithm

C PSLG bounding the domain Ω to be meshed.
T current triangulation
$T_{|\Omega} = T \cap \Omega$
constrained edges : subedges of edges of C

- Initiallisation $T =$ constrained Delaunay triangulation of C
- Refinement Apply one of the following rules,

with priority according to index, until no one applies

1. if there is an encroached constrained edge e,
   refine-edge(e) i.e.
   insert $c = \mathrm{midpoint(e)}$ in T
2. if there is a bad facet $f \in T_{|\Omega}$,
   conditionally-refine-facet(f) i.e.:
   $c =$ circumcenter(f)
   if c encroaches a constrained edge e, refine-edge(e).
   else insert(c) in T

The Delaunay refinement theorem

Theorem (Ruppert 95 - Shewchuk 98)

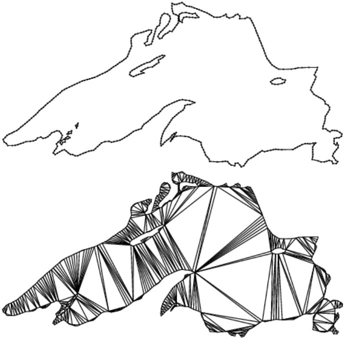
The Delaunay refinement algorithm ends provided that :

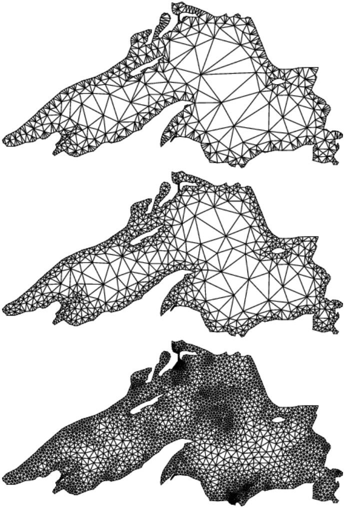
- *the size condition is an upper bound on triangles circumradii*
- *the shape condition is an upper bound* $B \geq \sqrt{2}$ *on radius-edge ratio of triangles*
- *adjacent PSLG edges (i. e. PSLG edges sharing a vertex) do not form angles smaller than* 60^o

The resulting mesh has no triangle with an angle less than $\arcsin \frac{1}{2B}$ $(= 20,7^o$ *for* $B = \sqrt{2})$

Example of 2D meshes

generated by Delaunay refinement





bounds on α
15^o, 25.6^o, 34.2^o respectively

Proof of Delaunay refinement theorem

Main idea

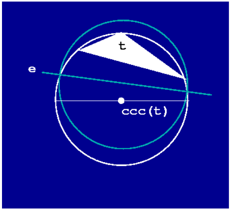
Use a volume argument
to bound the number of added (Steiner) vertices

Lemma (Steiner vertices)

Any Steiner vertex is inside or on the boundary of the domain Ω to be meshed

Proof.

If the circumcenter $cc(t)$ of triangle t
is not inside Ω,
some constrained edge e of T
is encroached by the vertices of t.



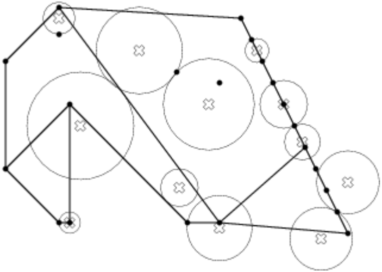
□

Local feature size

Definition (Local feature size)

Given a PSLG C and a point p,
the local feature size $\mathrm{lfs}(p)$ of p is the radius of the smallest disk centered in p and intersecting two disjoint elements of C, i.e.

- either two vertices of C
- or an edge and a non incident vertex
- or two disjoint edges of C.



$\mathrm{lfs}()$ is a Lipschitz function

$$\mathrm{lfs}(u) \leq \mathrm{lfs}(v) + \|uv\|$$

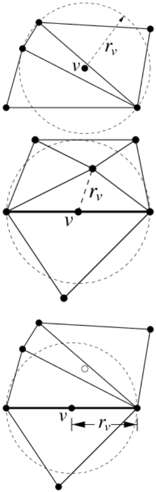
Insertion radius

Definition (Insertion radius)

rejected vertex = circumcenter considered for insertion and rejected for encroachment

v is a vertex of T or a rejected vertex.
The insertion radius r_v of v is the length of the smallest edge incident to v
right after insertion of v if v is inserted in T.

- v is a vertex of PSLG C
  r_v = distance to nearest visible vertex in C.
- $v = \mathrm{circumcircle}(t)$, ($v$ inserted or rejected)
  $r_v = \mathrm{circumradius}(t)$
- $v \in$ edge e encroached by p
  $r_v = \|e\|/2$ if p rejected
  r_v = distance to closest encroaching vertex otherwise

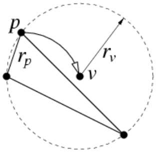


Parent vertex

Definition (Parent vertex)

Each added or rejected vertex v
is associated a parent vertex p.

- v is a vertex of PSLG C, no parent.
- $v = \mathrm{circumcircle}(t)$, ($v$ inserted or rejected)
  p is the vertex of the smallest edge of t
  that has been inserted last.
- v inserted in an encroached edge e
  p is the encroaching vertex closest to v
  (p may be a vertex of T or a rejected vertex.)



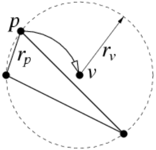
Insertion radius lemma

Lemma (Insertion radius)

For any vertex v, inserted in the mesh or rejected, either $r_v \geq \text{lfs}(v)$ or $r_v \geq Cr_p$, where p is the parent of v, with :

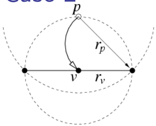
1. $C = B$ *if v is circumcenter*
2. $C = 1/\sqrt{2}$
   if $v \in$ a PSLG edge and p is rejected
3. *if v and p are in PSLG edges*
   - *the two edges are disjoint* $r_v \geq \text{lfs}(v)$
   - *the two edges form an angle* α
     $C = 1/2 \cos\alpha$ *if* $\alpha \in [45^o, 90^o]$
     $C = \sin\alpha$ *if* $\alpha \leq 45^o$

Case 1



$r_p \leq l_{min}(t)$
$r_v \geq Bl_{min}(t)$

Case 2

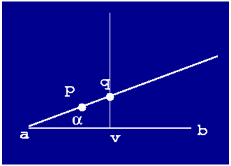


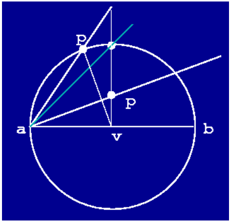
$r_p \leq min(\|pa\|, \|pb\|$
$\leq \sqrt{2}r_v$

Insertion radius lemma

Case 3

v and p are in PSLG edges
both edges share a vertex and form angle α





$r_v = \|pv\|$ $\qquad \frac{r_v}{r_p} \geq \frac{\|pv\|}{\|ap\|}$
$r_p \leq \|ap\|$
$\|pv\|^2 = \|ap\|^2 + \|av\|^2 - 2\|ap\|\|av\| \cos\alpha$
$\frac{\|pv\|^2}{\|ap\|^2} = 1 + \frac{\|av\|^2}{\|ap\|^2} - 2\frac{\|av\|}{\|ap\|} \cos\alpha$ $\qquad$ minimum $= \sin^2\alpha$ for $\frac{\|av\|}{\|ap\|} = \cos\alpha$
but p is in smallest circumcircle of edge $e = ab$

Proof of Delaunay refinement theorem

lfs_{min} = minimum distance between two disjoint elements of C
= $\min \mathrm{lfs}(p)$ for $p \in$ vertices of C

Lemma (Lower bound on edge length)

If the PSLG C has no pair of adjacent edges forming an angle less than 60^o, if there is no size condition, and if the upper bound on radius-edge ratio is $B \geq \sqrt{2}$, the Delaunay refinement produces no edge in T smaller than lfs_{min}.

End of Delaunay refinement theorem proof

T is a Delaunay triangulation
with no edge shorter than lfs_{min}
$\Longrightarrow$ the disks around each vertex with radius $\mathrm{lfs}_{min}/2$
do not intersect.

n = number of vertices in the mesh

$$n \frac{1}{6} \pi \frac{\mathrm{lfs}_{min}^2}{4} \leq \mathrm{area}(\Omega)$$

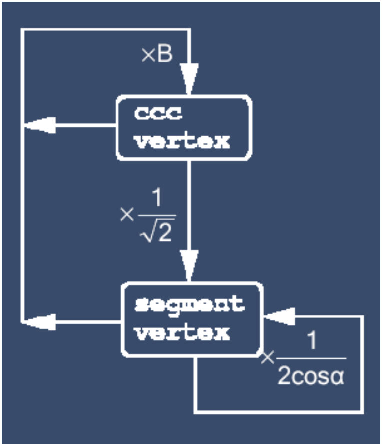
Proof of the lower bound on edge length

Assume the lower bound holds until the insertion of vertex v in T
For any ancestor q of v, $r_q \geq \mathrm{lfs}_{min}$.
p the parent of v
g the parent of p

- v is a circumcenter, $r_v \geq Br_p$
- v in a PSLG edge, p rejected

$$r_v \geq \frac{r_p}{\sqrt{2}} \geq \frac{Br_g}{\sqrt{2}} \geq r_g$$

- v and p in a PSLG edge
  - disjoint edges
    $r_v \geq \mathrm{lfs}(v) \geq \mathrm{lfs}_{min}$
  - edges forming angle $\alpha \geq 60^o$
    $r_v \geq \frac{r_p}{2\cos\alpha} \geq r_p$



weighted density $d(v) = \frac{\text{lfs}(v)}{r_v}$

Lemma (Weighted density lemma 1)

For any vertex v with parent p, if $r_v \geq Cr_p$, $d(v) \leq 1 + \frac{d(p)}{C}$

Proof.

$$\text{lfs}(v) \leq \text{lfs}(p) + \|pv\| \leq \text{lfs}(p) + r_v \leq d(p)r_p + r_v \leq \left(\frac{d(p)}{C} + 1\right) r_v \qquad \square$$

Lemma (Weighted density lemma 2)

There are constants $D_e \geq D_f \geq 1$ such that :
for any circumcenter v, inserted or rejected, $d(v) \leq D_f$
for any vertex v inserted in a PLSG edge, $d(v) \leq D_e$.
Thus, for any vertex of the mesh $d(v) \leq D_e \iff r_v \geq \frac{\text{lfs}(v)}{D_e}$.

Proof of weighted density lemma 2

Assume that lemma is true up to the insertion of vertex v
p parent of v

- v is a circumcenter
  $r_v \geq Br_p \Longrightarrow d(v) \leq 1 + \frac{d(p)}{B}$ assume $1 + \frac{D_e}{B} \leq D_f$ (1)
- v is on a PSLG edge e
  - p is a PSLG vertex
    or $p \in$ PSLG edge e', $e \cap e' = \emptyset$
    $r_v = \mathrm{lfs}(v) \Longrightarrow d(v) \leq 1$
  - p is a rejected circumcenter
    $r_v \geq \frac{r_p}{\sqrt{2}} \Longrightarrow d(v) \leq 1 + \sqrt{2}d(p)$
    assume $1 + \sqrt{2}D_f \leq D_e$ (2)
  - $p \in$ PSLG edge e', e and e' form angle α
    $r_v \geq \frac{r_p}{2\cos\alpha} \Longrightarrow d(v) \leq 1 + 2\cos\alpha d(p)$
    assume $1 + 2\cos\alpha_{min} D_e \leq D_e$ (3)

Proof of weigthed densiy lemma (end)

There are $D_e \geq D_f \geq 1$ such that :

$1 + \frac{D_e}{B} \leq D_f$ (1)

$1 + \sqrt{2} D_f \leq D_e$ (2)

$1 + 2\cos\alpha_{min} D_e \leq D_e$ (3)

$(2) \implies D_f \leq D_e$

$(1) + (2) \implies 1 + \frac{D_e}{B} \leq D_f \leq \frac{D_e - 1}{\sqrt{2}}$

$(3) \implies D_e \geq \frac{1}{1 - 2\cos\alpha_{min}}$

$$D_e \geq \max\left(\frac{(1+\sqrt{2})B}{B-\sqrt{2}}, \frac{1}{1-2\cos\alpha_{min}}\right)$$

$$D_f = 1 + \frac{D_e}{B}$$

Theorem (A relative bound on edge length)

Any edge of the mesh incident to vertex v has length l st : $l \geq \frac{\mathrm{lfs}(v)}{D_e+1}$

Proof.

Edge vw

- if w is inserted before v, $\|vw\| \geq r_v \geq \mathrm{lfs}(v)/D_e$
- else, $\|vw\| \geq r_w \geq \mathrm{lfs}(w)/D_e \geq (\mathrm{lfs}(v) - \|vw\|)/D_e$

□

Upper bound on the number n of vertices of the mesh

For any vertex v, disc $\Sigma(v) = (v, \rho(v))$ with $\rho(v) = \frac{\mathrm{lfs}(v)}{2(D_e+1)}$

$$\int_\Omega \frac{dx}{\mathrm{lfs}(x)^2} \geq \sum_v \int_{\Sigma(v)\cap\Omega} \frac{dx}{\mathrm{lfs}(x)^2} \geq \sum_v \frac{1}{6} \frac{\pi\rho(v)^2}{(\mathrm{lfs}(v)+\rho(v))^2} \geq \frac{n}{6} \frac{\pi}{(3+2D_e)^2}$$

Lower bound on number of mesh vertices

Theorem (Lower bound on the mesh size)

Any mesh with minimum angle α of a domain Ω has a number n of vertices such that

$$n \geq \frac{1}{3c^2\pi} \int_{\Omega} \frac{dx}{\mathrm{lfs}(x)^2},$$

where the constant c depends on the minimum angle α.

Lemma (Edge length ratio 1)

Edge length ratios in a mesh with minimum angle α
between two edges of the same triangle $\frac{lb}{la} \leq \frac{1}{\sin\alpha}$
two edges incident to the same vertex $\frac{lb}{la} \leq \left(\frac{1}{\sin\alpha}\right)^{\frac{2\pi}{\alpha}}$

Lower bound on number of mesh vertices

Definition

$T(\Omega)$ a mesh of domain Ω

point $p \in \Omega$, $\mathrm{lm}(p) =$ length of the longest edge of $t \in T(\Omega)$ including p

Lemma (Longest edge lemma 1)

If $T(\Omega)$ has minimum angle α

for p and q in adjacent triangles, $\mathrm{lm}(q) \leq \frac{1}{\sin\alpha}\mathrm{lm}(p)$

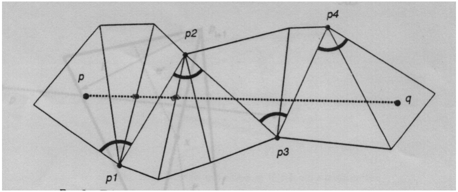
for any p and q in Ω, $\mathrm{lm}(q) \leq c_1 lm(p) + c_2\|pq\|$

with

$c_1 = \left(\frac{1}{\sin\alpha}\right)^{\lfloor\frac{\pi}{\alpha}\rfloor+2}$

$c_2 = 4\left(\frac{1}{\sin\alpha}\right)^{\lfloor\frac{\pi}{\alpha}\rfloor+2}$

Proof of longest edge lemma 1



A fan : a set of consecutive triangles crossed by pq
with two vertices on the same side of pq

A fan has at most $K = \left\lfloor \frac{\pi}{\alpha} \right\rfloor$ triangles

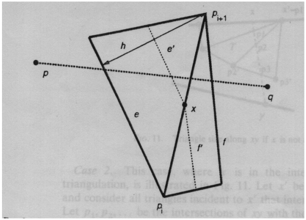
k number of edges intersected by pq

if $k \leq K + 3$, $\mathrm{Im}(q) \leq \mathrm{Im}(p) \left(\frac{1}{\sin \alpha}\right)^{K+2}$

if $k > K + 3$, pq intersects at least one transition edge
$p_i p_{i+1}$ between two fans

Proof of longest edge lemma 1 (end)

$p_i p_{i+1}$ last transition edge crossed by pq



$t = t_i$ or t_{i+1} depending on midpoint of p_i, p_{i+1} wrt pq

h elevation of t, p' point in t
$\|pq\| \geq \frac{h}{2}$

$$\mathrm{lm}(p') \leq \left(\frac{2}{\sin\alpha}\right) h \leq \left(\frac{4}{\sin\alpha}\right) \|pq\|$$

$$\mathrm{lm}(q) \leq \|pq\| \left(\frac{4}{\sin\alpha}\right) \left(\frac{1}{\sin\alpha}\right)^{K+1}$$

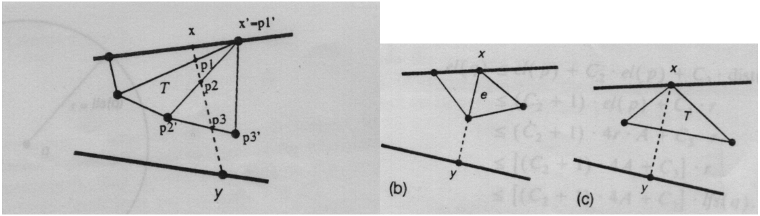
Lower bound on nb of mesh vertices

Lemma (Longest edge lemma 2)

If x and y are two points on disjoint edges of the PSLG, there is a point q in xy with $\mathrm{lm}(q) \leq \left(\frac{4}{\sin\alpha}\right) \|xy\|$

Proof.

Easy if x or y are vertices.
Transition edge analogous to the previous slide otherwise.

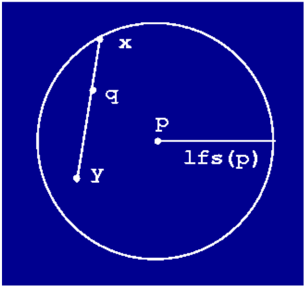


□

Lower bound on nb of mesh vertices (end)

Lemma (Longest edge lemma 3)

If $T(\Omega)$ has minimum angle α, for any $p \in \Omega$, $\mathrm{lm}(p) \leq c_3 \mathrm{lfs}(p)$



Proof.

Disc $\Sigma(p, \mathrm{lfs}(p))$
$\mathrm{lm}(q) \leq \frac{4}{\sin\alpha} \|xy\| \leq \frac{4}{\sin\alpha} 2\mathrm{lfs}(p)$
$\mathrm{lm}(p) \leq c_1 \mathrm{lm}(q) + c_2 \|pq\|$
$\mathrm{lm}(p) \leq \left(\frac{8c_1}{\sin\alpha} + c_2\right) \mathrm{lfs}(p) \leq c_3 \mathrm{lfs}(p)$

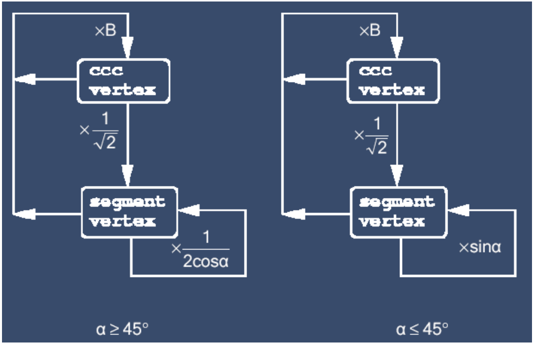
□

Lower bound on nb of mesh vertices

$$\int_\Omega \frac{dx}{\mathrm{lfs}(x)^2} \leq \sum_{t \in T(\Omega)} \int_t \frac{dx}{\mathrm{lfs}(x)^2} \leq c_3 \sum_{t \in T(\Omega)} \int_t \frac{dx}{\mathrm{lm}(x)^2} \leq c_3 n \frac{\sqrt{3}}{4}$$

Delaunay refinement and small input angles

Small angles between edges of the input PSLG cause problem for the Delaunay refinement algorithm. Lower bound on insertion radii no longer holds.



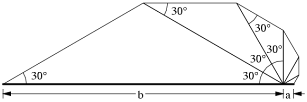
Theorem (Negative result)

Whatever may be the lower bound θ for the angles, there are PSLG which cannot be triangulated without creating new angles less than θ

Lemma (Edge ratio 2)

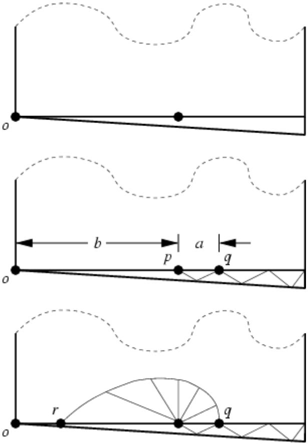
If all mesh angles $> \theta$, successive mesh edges on the same PSLG edge have a bounded length ratio.

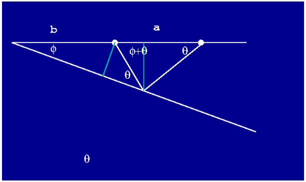
$\frac{lb}{la} \leq \left(\frac{1}{\sin\theta}\right)^{\frac{\pi}{\theta}}$ $\qquad$ $\frac{lb}{la} \leq (2\cos(\theta))^{\frac{\pi}{\theta}}$



Delaunay refinement and small input angles

Proof of negative result





$\frac{la}{lb} \leq B_2 = \frac{\sin\phi}{\sin\theta}\left(\cos(\theta+\phi) + \frac{\sin(\theta+\phi)}{\tan\theta}\right)$

$\frac{lb}{la} \leq B_2 = (2\cos(\theta))^{\frac{\pi}{\theta}}$

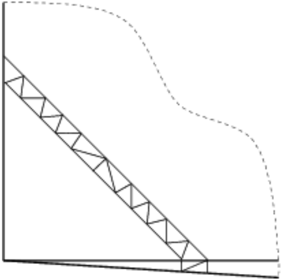
If $B_1 B_2 < 1$, the mesh has another vertex r between o and p.

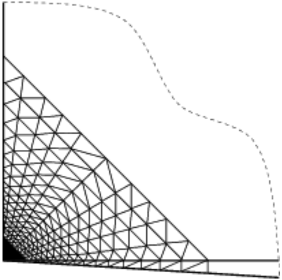
The same situetion occurs at r

$\Longrightarrow$ no finite mesh is possible

Delaunay refinement and small input angles

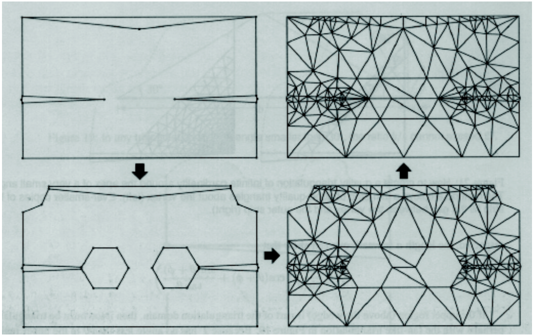
...unless having an infinite number of triangles





Meshing domain with small input angles

Corner looping

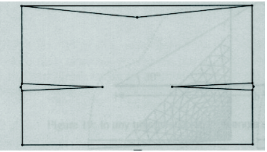


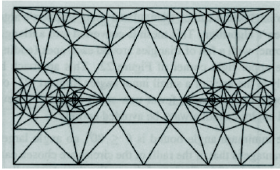
Advantages : new small angles appear only at PSLG vertices wit small input angles

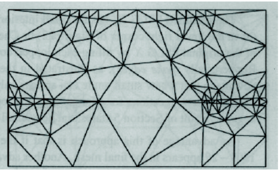
Drawbacks : reduces lfs

Meshing domain with small input angles

Terminator





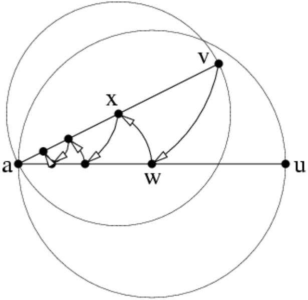


Corner looping

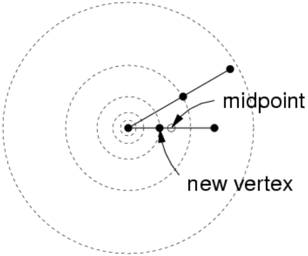
Terminator

Meshing domain with small input angles

Pb 1 : direct coupling
input angle $< 45^O$

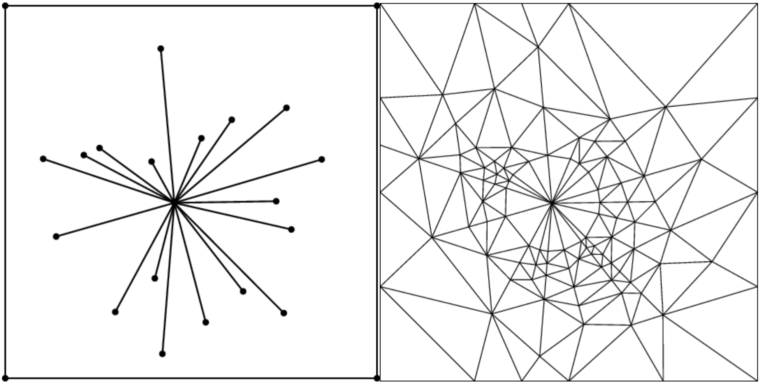


Solution : refine edges incident
to small angles
along concentric circles



Meshing domain with small input angles

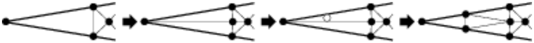
Example using concentric shell refinement



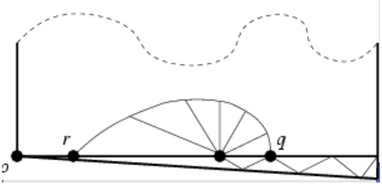
Meshing domain with small input angles

Concentric circles refinement does not enough

Concentric circles solve the mesh problem for polygonal region

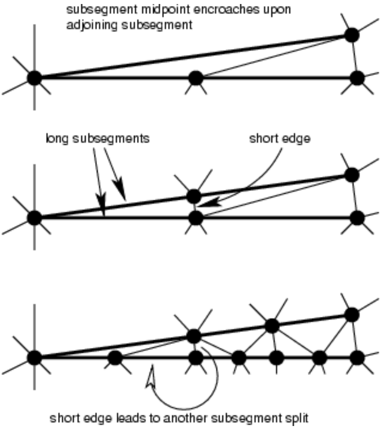


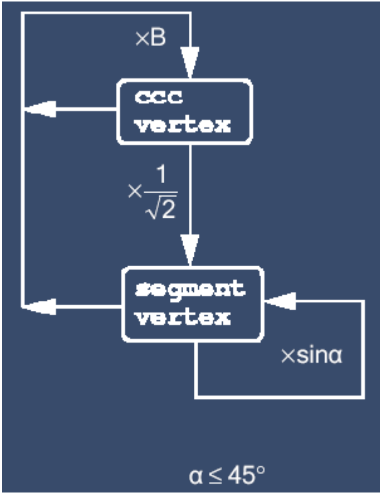
Concentric circles does not solve indirect coupling



Meshing domain with small input angles

Pb 2 : indirect coupling





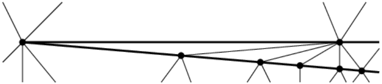
Meshing domain with small input angles

A first solution to the indirect coupling problem

Refuse the insertion of a vertex whose insertion radius is smaller than one of his ancestors

Drawback

- The final mesh is not guaranteed to be Delaunay
- The mesh includes large angles



Meshing domain with small input angles

Terminator algorithm [Shewchuk 2000]

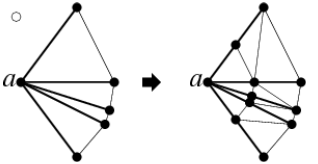
Terminator meshing :

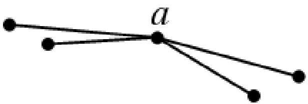
Delaunay refinement meshing with two additionnal rules

Additionnal rule 1

Refine edges in clusters along concentric circles

A cluster : a set of constrained edges incident to the same vertex and forming angles smaller than 60^o





Meshing domain with small input angles

Terminator algorithm

Additionnal rule 2

Refuse refinement of bad triangles that reduce insertion radius

More precisely :
t bad triangle whose circumcenter p encroaches edge e in a cluster,
v the refinement point on e,
$r_{min}(v)$ smallest insertion radius in the cluster if v is inserted,
g the parent of p.

refinement of cluster edge e is agreed in the following cases :

A. $r_{min}(v) \geq r_g$
B. t does not satisfy the size criteria if any
C. the cluster is not yet reduced,
   i. e. all the cluster edges do not have the same length
D. (optional) there is no ancestor of v in the PSLG edge including e

If refinement of cluster edge e is refused,
t is kept in the mesh and will never be reconsidered for refinement.

Meshing domain with small input angles

Theorem (Terminator algorithm)

1. *The terminator algorithm ends*
2. *It provides a Delaunay mesh with no encroached constrained edge*
3. *Small angles occur in the mesh*
   only closed to small input angle.
   No angle is less than $\frac{\phi}{2\sqrt{2}}$
   where ϕ *is the smallest input angle.*

Proof of points 1 and 2.

A vertex in the mesh is a diminishing vertex if its insertion radius is smaller than one of it's ancestor insertion radius.
Only a finite number of diminishing vertices are inserted
(this happens in case B, C or D) □

Meshing domain with small input angles

Proof of Terminator algorithm theorem

Proof of point 3.

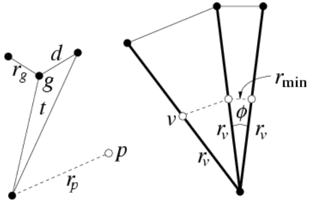
The circumcenter of any bad triangle t left in the mesh encroaches some edge in a cluster.
We show that the radius-edge ratio of t is : $\rho \leq \frac{1}{\sqrt{2}\sin(\phi/2)}$.

notation as above
d length of smallest edge of t

$r_g \leq d \qquad r_p \leq \sqrt{2} r_v$

$r_{min} \leq r_g \qquad 2 r_v \sin\left(\frac{\phi}{2}\right) \leq r_{min}$

$$\rho = \frac{r_p}{d} \leq \frac{1}{\sqrt{2}\sin(\phi/2)}$$



□