ENVELOPPES CONVEXES, PSEUDOTRIANGULATIONS ET GRAPHES DE VISIBILITÉ DANS LES PLANS TOPOLOGIQUES PART I : TOPOLOGICAL PLANES ON $\mathbb{R}^2$ AND CHIROTOPES

Michel Pocchiola (Ens-Paris)/ Paris, Décembre 2006

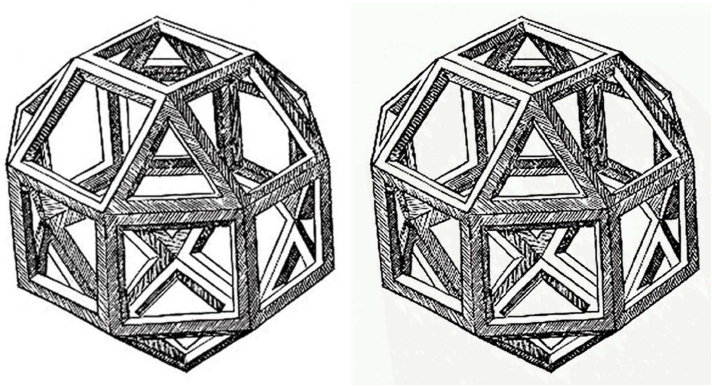


Figure 1: Première illustration d'un rhombicuboctaèdre en creux par Leonardo da Vinci pour la Divina proportione

Figure 2: Ritratto di Frà Luca Pacioli (1495). Luca Pacioli (1445 - 1517) is the central figure in this painting exhibited in the Museo e Gallerie di Capodimonte in Napoli (Italy). The painter is unknown, although some people are convinced the painter is Jacopo de' Barbari (1440-1515). Table is filled with geomerical tools: slate, chalk, compas, a dodecahedron model and a rhombicuboctahedron half-filed with water is hanging in the air. Pacioli is demonstrating a theorem by Euclid.

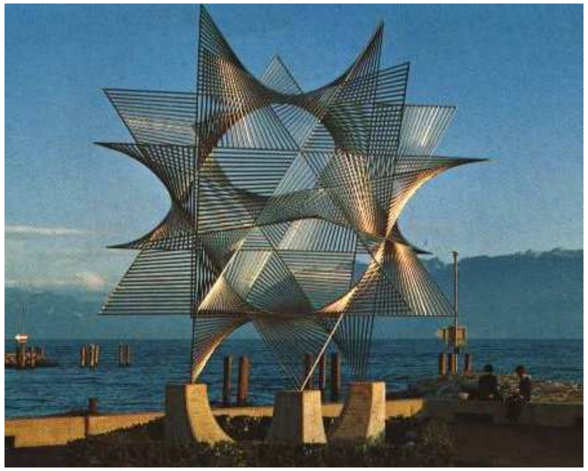
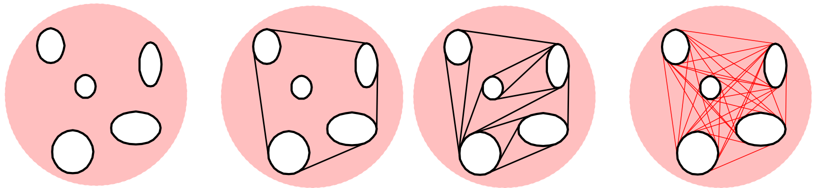


Figure 3: Sculpture d'Angel DUARTE, Lausanne, Suisse.

INTRODUCTION



Theorem 1 (Habert & P. 04/06, Angelier & P. 03, P. & Vegter 96). *The convex hull, a (greedy) pseudotriangulation, the visibility graph of a finite family of pairwise disjoint convex bodies of a topological plane on* $\mathbb{R}^2$ *are each computable using only* $O($*size of the output* $+$ *size of the input* $\times \log$ *size of the input*$)$ *probes of its chirotope.* □

INCIDENCE STRUCTURE/PROJECTIVE PLANES

An *incidence structure* is a triple (P, L, F) of sets with $F \subseteq P \times L$. The elements of P are termed points, the elements of L are termed lines, and the elements of F are termed flags or incidences. A set of points is termed collinear if there exists a line incident with all these points.

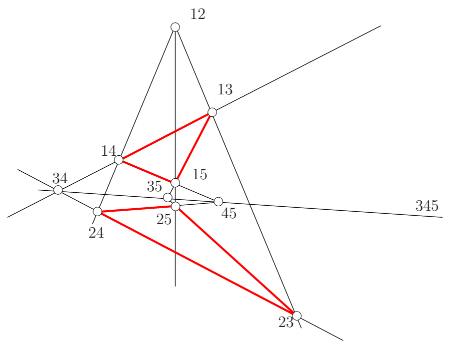
Definition 1. *A projective plane $\mathcal{P}$ is an incidence structure (P, L, F) which satisfies the three following conditions*

1. *any two distinct points are incident with a unique line, their joining line;*
2. *any two distinct lines are incident with a unique point, their intersection point;*
3. *there exists a non-degenerate quadrangle, i.e., a set of four points no three of which are collinear.*

Often (but not always) the lines are identified with their sets of incident points. In that case we write $p \in l$ instead of $(p, l) \in F$. The pencil L_p of a point p is the set of its incident lines.

Classical examples. Let K be a not necessarily commutative field, and consider K^3 as a right vector space over K. Denote by P the set of all one-dimensional subspaces of K^3, and by L the set of all two-dimensional subspaces. Then $\mathcal{P}_2(K) = (P, L, \subseteq)$, with inclusion $\subseteq$ as incidence, is a projective plane, *the projective plane over* K.

THE DESARGUES CONFIGURATION



The Desargues configuration is the incidence structure

$$\mathcal{D} = (\binom{\Omega}{2}, \binom{\Omega}{3}, \subseteq),$$

where $\Omega = \{1, 2, 3, 4, 5\}$.

DESARGUESIAN PLANES

A projective plane $\mathcal{P}$ is said to be *desarguesian*, if the Desargues configuration 'always closes' in $\mathcal{P}$; this means that the following 'configurational' proposition holds: label ten distinct points and ten distinct lines of $\mathcal{P}$ by the elements of $\binom{\Omega}{2}$ and $\binom{\Omega}{3}$, respectively, and assume that 29 of the 30 incidences of $\mathcal{D}$ hold in P. Then also the remaining incidence of $\mathcal{D}$ holds in $\mathcal{P}$ (it does not matter which incidence is singled out as the conclusion, because $\mathcal{D}$ has a flag-transitive collineation group).

Theorem 2 (Hilbert, 1899)**.** *The desarguesian projective planes are precisely the planes $\mathcal{P}_2 K$ where K is a (not necessarily commutative) field.* □

[1] Burkard Polster and Günter Steinke. Geometries on Surfaces. Number 84 in Encyclopedia of Mathematics and its applications. Cambridge, 2001.

[2] Helmut Salzmann, Dieter Betten, Theo Grundhöfer, Hermann Hähl, Rainer Löwen, and Markus Stroppel. Compact projective planes. Number 21 in De Gruyter expositions in mathematics. Walter de Gruyter, 1995.

AFFINE PLANES AND PROJECTIVE COMPLETION

Definition 2. *An affine plane $\mathcal{A}$ is an incidence structure (A, L, F) that satisfies the three following conditions*

1. *any two distinct points are incident with a unique line;*
2. *for every pair $(p, l) \in A \times L$ there exists a unique line k of the pencil of p such that either $k = l$ or k and l have no point in common; we call k the parallel to l through p;*
3. *there exists a triangle in $\mathcal{A}$, i.e., a set of three points which are not collinear.*

Let $\mathcal{P} = (P, L, \in)$ be a projective plane and let W be an arbitrary line of $\mathcal{P}$. Then the *affine part*

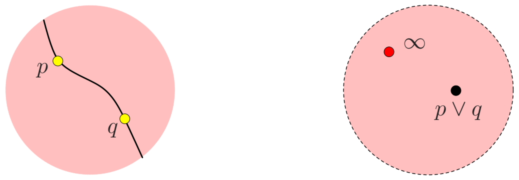
$$\mathcal{P}^W = (P \setminus W, L \setminus \{W\}, \in)$$

is an affine plane. Conversely, every affine plane $\mathcal{A} = (A, L, F)$ has a *projective completion* $\overline{\mathcal{A}}$, which is the projective plane obtained from $\mathcal{A}$ by adding a new line W (called the line at infinity) and a set of new points on W (called the points at infinity) which correspond to the parallel classes of $\mathcal{A}$. (Hence every affine plane has the form $\mathcal{P}^W$.) More precisely we set

$$\begin{aligned} \overline{A} &= A \cup [L], \\ \overline{L} &= \{l \cup [l] \mid l \in L\} \cup \{[L]\}. \end{aligned}$$

where $[l]$ denotes the parallel class of $l \in L$.

$\mathbb{R}^2$-PLANES (TOPOLOGICAL PLANES ON $\mathbb{R}^2$)



space of points $\approx \mathbb{R}^2$ space of lines $\approx \mathbb{P}^2 \setminus \{\infty\}$

Definition 3. *A pseudoline of $\mathbb{R}^2$ is the image of $\mathbb{R} \times \{0\}$ by an homeomorphism of $\mathbb{R}^2$.* □

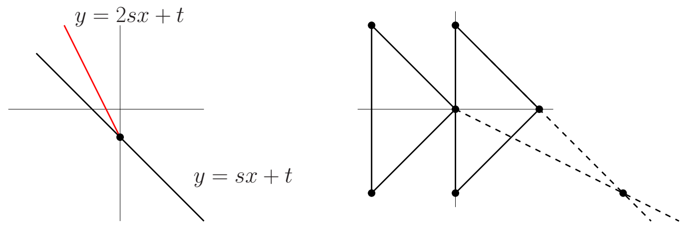
Definition 4. *An $\mathbb{R}^2$-plane is a point-line incidence structure whose point set X is a topological space homeomorphic to $\mathbb{R}^2$ and whose line set satisfies the following two conditions :*
(1) Each line is a pseudoline of X;
(2) Any two distinct points are contained in exactly one line, their joining line. □

Definition 5. *An affine $\mathbb{R}^2$-plane is a $\mathbb{R}^2$-plane such that for every point-line pair (p, l) there exists a unique line k through p such that either $k = l$ or k and l are disjoint; the line k is called the parallel to l through p.* □

EXAMPLES

1. the *real affine plane* : for lines we take the sets $L_{s,t} = \{(x, y) \in \mathbb{R}^2 \mid y = sx + t\}$, $s, t \in \mathbb{R}$ and all "verticals" $\{x\} \times \mathbb{R}$, $x \in \mathbb{R}$. The parabola model of the real affine plane : as lines we take all parabolas $Q_{a,b} = \{(x, y) \mid y - b = (x - a)^2\}$ with $a, b \in \mathbb{R}$ and all verticals.
2. For $d > 1$ fixed we take all verticals and all curves $y - b = |x - a|^d$ with $a, b \in \mathbb{R}$ These examples are affine planes which are not desarguesian for $d \neq 2$.
3. the *real hyperbolic plane* is the restriction of the real affine plane to the open unit disk (due to F. Klein, 1871). The restriction of a topological plane to any of its non-empty convex open subsets is again a topological plane.
4. *The affine Moulton planes* $\mathcal{M}_k, k > 1$. We start with the real affine plane and we replace the lines $L_{s,t} = \{(x, sx + t)\}$ with negative slope s by the kinked lines
$$L^*_{s,t} = \{(x, sx + t) \mid x \geq 0\} \cup \{(x, skx + t) \mid x \leq 0\},$$
where $k > 1$ is fixed. $\mathcal{M}_k$ is not desarguesian.
5. *Shift planes*. Vertical lines plus a curve $\{(x, y \in \mathbb{R}^2 \mid y = f(x)\}$ and its images under the shift group $\{(x, y) \mapsto (x + a, y + b) \mid a, b \in \mathbb{R}^2\}$. (For example $f : \mathbb{R} \to \mathbb{R}$ differentiable such that the derivative is a homemorphism.)
6. *Arc planes*. Replace the lines with positive slope by the shifted arcs $\{(x, y) \in \mathbb{R}^2 \mid y - n = e^{x-m}\}$, $m, n \in \mathbb{R}$. No embedding into any affine topological plane.

MOULTON PLANES



FIRST TOPOLOGICAL PROPERTIES

Open/closed half-planes, open/closed intervals, convex sets, Pasch Axiom, neighbourhood basis, collinearity and ordre of collinearity under limits.

NATURAL TOPOLOGY ON THE SET OF LINES

Convergence au sens de Hausdorff. Soit l_i une suite de droites, $\liminf l_i$ l'ensemble des limites des suites convergentes de points p_i, $p_i \in l_i$, et $\limsup l_i$ l'ensemble des points d'accumulation des suites q_i, $q_i \in l_i$. On dit que l_i converge au sens de Hausdorff vers $l \in L$, si $l = \liminf l_i = \limsup l_i$.

La *topologie naturelle* sur L est définie comme suit: $F \subseteq L$ est un fermé de H si et seulement si F contient la limite de Hausdorff de toute suite de F convergente au sens de Hausdorff.

Topologie finale, de jointure, d'intersection.

Theorem 3. *Les topologies naturelle, finale, de jointure et d'intersection coincident. Muni de cette topologie L est un espace séparé, et la convergence est équivalente à la convergence au sens de Hausdorff. De plus l'application de jointure est continue et ouverte.* □

Theorem 4. *La topologie naturelle sur L est l'unique topologie telle que l'application de jointure soit continue et l'application d'intersection soit stable (i.e., continue et de domaine un ouvert de L^2)* □

Theorem 5. *L'ensemble des droites munie de la topologie naturelle est une surface homéomorphe à un ruban de Möbius et le faisceau des droites passant par un point est une pseudodroite. (L'ensemble des droites orientées est un cylindre.)* □

PROJECTIVE COMPLETION

Definition 6. *Let $\mathcal{A} = (A, L)$ be an affine topological plane. The projective completion of (A, L) is defined as the point-line incidence geometry $(\overline{A}, \overline{L})$ where $\overline{L}$ is endowed with the "one-point compactification" topology $\hat{H}$ and $\overline{A}$ is endowed with the "pencil intersection" topology PI (i.e. la topologie de base d'ouverts les $J_p \vee J_q$ où p et q sont deux points de $\overline{\mathcal{A}}$ et J_p et J_q sont deux intervalles ouverts disjoints des faisceaux de droites passant par p et q, respectivement).* □

Theorem 6. *$\overline{A}$ and $\overline{L}$ induce the natural topologies on A and L. Furthermore $\overline{\mathcal{A}}$ is a topological projective plane (i.e., que les applications de jointure et d'intersection sont continues). Each affine part of $\overline{\mathcal{A}}$ is an affine topological plane. Les topologies $\hat{H}$ et PI sont compactes.* □

CHARACTERIZATION

Proposition 7. *Chaque droite et chaque faisceau de droites de la complétude projective d'un plan affine topologique est homéomorphe à $\mathbb{S}^1$, et les espaces de points et de droites sont homéomorphes au plan projectif réel $P_2(\mathbb{R})$. De plus l'espace des incidences $F = \{(p, l) \mid p \in l\} \subset \overline{A} \times \overline{L}$ est une variété connexe de dimension 3i homéomorphe à l'espace des incidences de $P_2(\mathbb{R})$.*

Corollary 8. *Soit (P, L) la complétude projective d'un plan affine topologique et soit $p \in P$ et $l \in L$. Alors les deux espaces $P \setminus \{p\}$ et $L \setminus \{l\}$ sont homéomorphes au ruban de Möbius.* □

Theorem 9. *Soit $\mathcal{P} = (P, L)$ un plan projectif dont l'espace des points est un espace topologique homéomorphe au plan projectif réel $P_2(\mathbb{R})$ tel que chaque droite soit homéomorphe à $\mathbb{S}^1$. Alors pour toute droite $W \in L$, la partie affine $\mathcal{L}^W$ de $\mathcal{P}$ est un plan topologique, et $\mathcal{P}$ est la complétion projective de $\mathcal{P}^W$. En particulier $\mathcal{P}$ est un plan projectif topologique.* □

NOTES

ARRANGEMENTS OF PSEUDOLINES IN A MÖBIUS STRIP

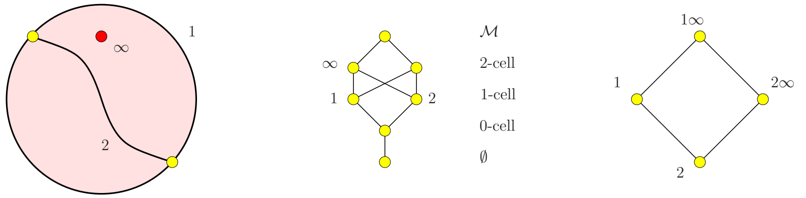
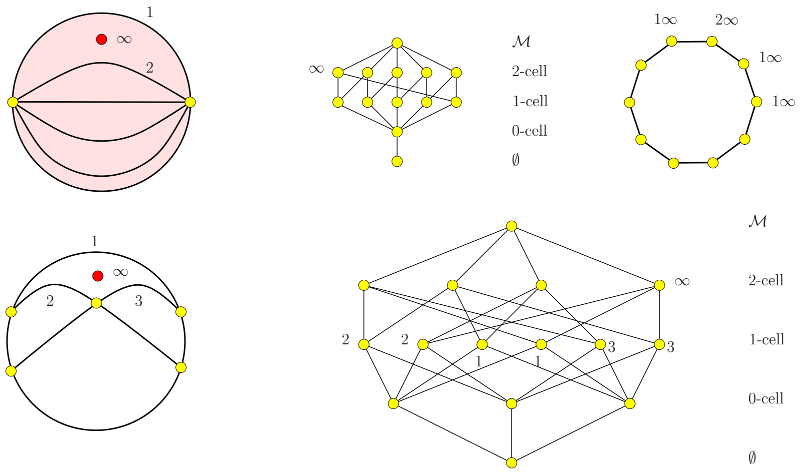


Figure 4: An arrangement of two pseudolines indexed by $\{1, 2\}$, the (labeled) Hasse diagram of its cell lattice, and adjacency graph of its (labeled) flags.

Definition 7. *Let $\mathcal{M}$ be a Möbius strip. A pseudoline in $\mathcal{M}$ is the range of a simple loop generator of $\pi_1(\mathcal{M})$, and an arrangement of pseudolines in $\mathcal{M}$ is a finite family of pseudolines in $\mathcal{M}$ that cross pairwise in exactly one point. Induced cell complex, induced lattice, oriented arrangement, simple arrangement.* □

Extensively studied in the field of discrete and computational geometry: F. Levi (1926), G. Ringel (1956), B. Grünbaum (1972), J. E. Goodmann (1980), R. Cordovil (1982), J-P. Roudneff (1988), M.E. Mnëv (1988), P. Shor (1991), J.E. Goodmann & R. Pollack & R. Wenger and T. Zamfirescu (1994), Edelsbrunner & Guibas (1989), etc.

PENCILS AND CIRCLES



A pencil of pseudolines is an arrangement of pseudolines with nonempty intersection. A circle of pseudolines is an arrangement of pseudolines, with the property that each pseudoline contributes for one edge to the boundary of the unbounded cell.

ISOMORPHISM/ISOTOPY CLASSES

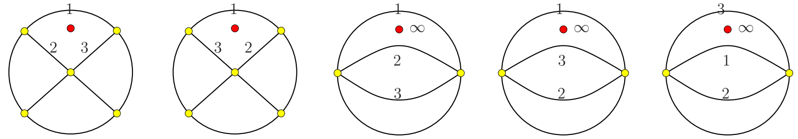
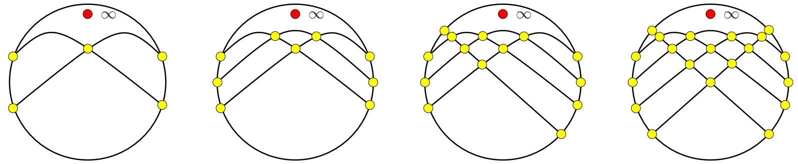


Figure 5: Representatives of the 5 isotopic classes of arrangements of three pseudolines defined on a given set of indices.

Definition 8. *Two arrangements are termed isomorphic if there is some homeomorphism of their underlying Möbius strips that maps the pseudolines in one arrangement to those in the other. Two arrangements are termed isotopic if there exists a homotopy of isomorphic arrangements connecting them.* □

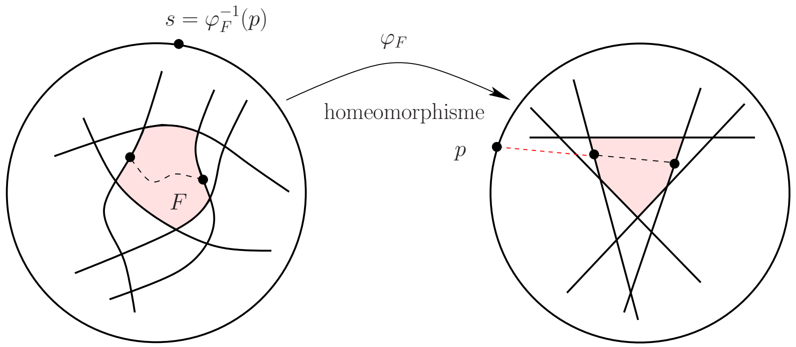
Theorem 10. *Two arrangements are isomorphic if and only if their induced lattices are isomorphic. Two arrangements (defined on the same Möbius strip) are isotopic if and only their induced oriented lattices are isomorphic.* □

FOUR CIRCLES



The map that assigns to a circle of pseudolines the graph of unordered pairs of its indices whose corresponding edges on the boundary of the unbounded 2-cell are adjacent induces a one-to-one correspondance between the set of isomorphism classes of circles defined on a given set of indices and the set of circular graphs on that set of indices. Similarly the map that assigns to a pencil of pseudolines the graph of unordered pairs of indices whose corresponding pseudolines bound a bounded 2-cell induces a one-to-one correspondance between the set of isomorphism classes of pencils defined on a given set of indices and the set of linear graphs on that set of indices.

FROM PSEUDOLINES ARRANG. TO TOPOLOGICAL PLANES



Theorem 11 (J-E. Goodman et al, 1994)**.** *Every arrangement of pseudolines in the projective plane can be extended to a topological projective plane.* □

[1] J. E. Goodman, R. Pollack, R. Wenger, and T. Zamfirescu. Arrangements and topological planes. Amer. Math. Monthly, 101(9):866–878, November 1994.

RINGEL THEOREM

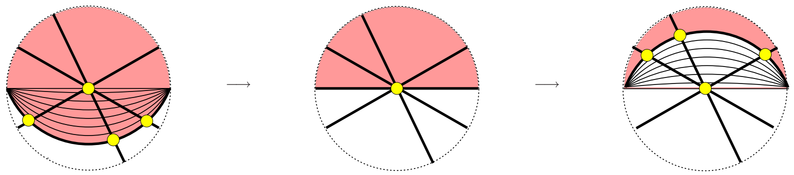
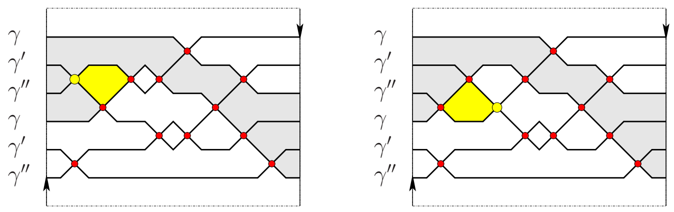


Figure 6: A mutation is the product of a merging half-mutation followed by its symmetric splitting half-mutation.

Theorem 12 (Ringel Theorem, 1956)**.** *Any pair of simple arrangements, defined on the same set of indices, are homotopic via a finite sequence of mutations followed by an isotopy.* □

Theorem 13. *Any arrangement of pseudolines is isomorphic to the dual arrangement of a planar finite family of points.* □

PROOF OF THEOREM 12



Essentially based on the following lemma

Lemma 14. *Let Γ be a simple arrangement of pseudolines and let $\gamma, \gamma' \in \Gamma$. Assume that there exists a vertex of $X(\Gamma)$ lying in the interior of the pinched Möbius band $M(\gamma, \gamma')$ bounded by γ and γ'. Then there exists a triangular 2-cell included in $M(\gamma, \gamma')$ with a side supported by γ' and a vertex included in the interior of $M(\gamma, \gamma')$.* □

CHIROTOPES OF ARRANGEMENTS OF PSEUDOLINES

Definition 9. *The chirotope of an arrangement is the map that assigns to each subset of indices of size at most three the isotopy class of the subarrangement indexed by this subset (note that two isotopic arrangements have the same chirotope).* □

Theorem 15. *The map that assigns to an isotopy class of arrangements its chirotope is one-to-one.* □

Theorem 16. *Let I be a finite set of indices and let $\mathcal{C}_k(I)$, $k \in \mathbb{N}$, be the set of maps that assign to each triple of I an isotopy class of arrangements of pseudolines indexed by that triple and whose restrictions to triples of subsets of indices of size at most k are chirotopes of arrangements of pseudolines. Then $\mathcal{C}_5(I) = \mathcal{C}_6(I) = \cdots = \mathcal{C}_\infty(I)$.*

CHIROTOPES OF FINITE PLANAR FAMILIES OF POINTS

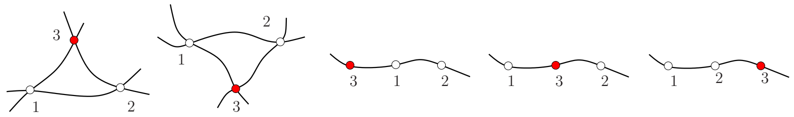


Figure 7: Representatives of the 5 chirotopes of plnar families of three points.

planar family of points = family of points of an oriented topological plane.

Definition 10. *The chirotope of a finite planar family of points is defined as the map that assigns to each ordered triple of indices of the family the* position vector *of the corresponding ordered triple of points, i.e., the vector of truth values (true or false) of the five relations «the third point of the triple lies on the open left side (open right side, initial/middle/terminal segment) of the directed line joining the first point of the triple to the second point of the triple».* □

Theorem 17. *The map that assigns to a finite planar family of points its dual arrangement induces a one-to-one correspondance between the set of chirotopes of planar family of points and the set of chirotopes of arrangements of pseudolines.* □

PRIMAL VERSION OF THEOREM 16

Theorem 18. *The set of chirotopes of families of points is exactly the set of maps defined on the set of triples of a finite set of indices whose restrictions to sets of triples of subsets of indices of size at most five are chirotopes of families of points.* □

NOTES

NOTES

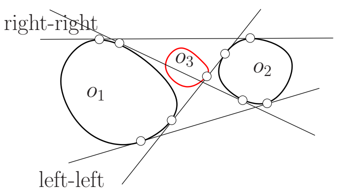
NOTES

NOTES

NOTES

NOTES

CHIROTOPES OF FINITE PLANAR FAMILIES OF PAIRWISE DISJOINT CONVEX BODIES



$(1,2,3)$	open left side	open right side	initial segment	middle segment	terminal segment
right-left	1	1	0	1	0
right-right	0	1	0	0	0
left-left	1	0	0	0	0
left-right	1	0	0	1	0

Definition 11. *The chirotope of a finite planar family of pairwise disjoint convex bodies is defined as the map that assigns to each ordered triple of indices of the family the position vector of the corresponding ordered triple of bodies, i.e., the vector of truth values (true or false) of the twenty relations «the third body of the triple intersects the open left side (open right side, initial/middle/terminal segment) of the right-left (right-right/left-left/left-right) directed double tangent joining the first body of the triple to the second body of the triple».* □

A FINITE CHARACTERIZATION

Let I be a finite set of indices, $k \in \mathbb{N}$, and $\mathcal{C}_k(I)$ be the set of $\varphi : I^3 \to \{0,1\}^{20}$ such that for all $J \in \binom{I}{\leq k}$ the restriction of φ to J^3 is a chirotope. Then

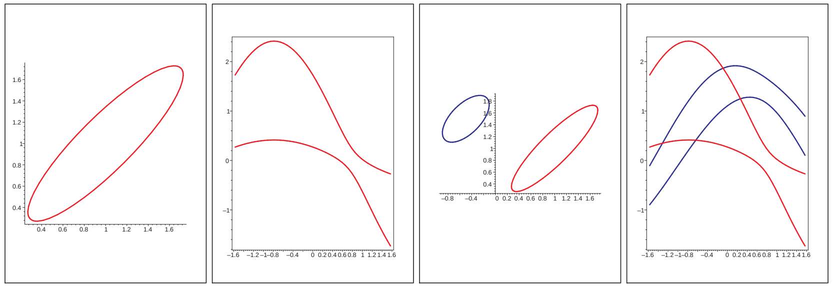
$$\mathcal{C}_5(I) = \mathcal{C}_6(I) = \cdots = \mathcal{C}_\infty(I).$$

Theorem 19 (Habert & P. 06). *The set of chirotopes of finite planar families of pairwise disjoint convex bodies is exactly the set of maps defined on the set of triples of a finite set of indices whose restrictions to sets of triples of subsets of indices of size at most five are chirotopes of planar families of pairwise disjoint convex bodies.* □

ARRANGEMENTS OF DOUBLE PSEUDOLINES.

DUAL OF A PLANAR CONVEX BODY

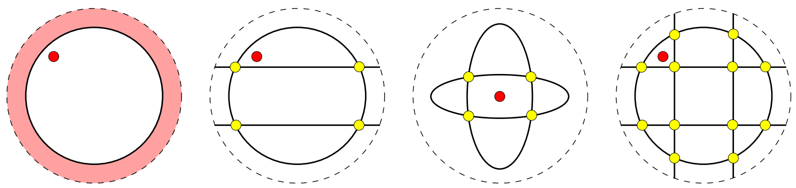
The dual of a convex body is defined as the set of tangents to that body. For example the dual



of the ellipse with center (x, y), half-axis a, b and angle φ of the real affine plane is the range of the curve that assigns to $t \in [0,1]$ the point $(2\pi t, -x \sin 2\pi t + y \cos 2\pi t - R(2\pi t - \varphi)) \in \mathbb{M}$ where $R(\theta) = (a^2 \sin^2\theta + b^2 \cos^2\theta)^{1/2}$ and $\mathbb{M} = \mathbb{R}^2/\{(x,y) \sim (x+\pi, -y)\}$.

Lemma 20. *The dual of a planar convex body is the range of a simple loop homotopic to the double of any generator of the fundamental group of the space of lines, and the duals of two pairwise disjoint convex bodies cross four times and induce a cell structure on the one-point compactification of the space of lines.* □

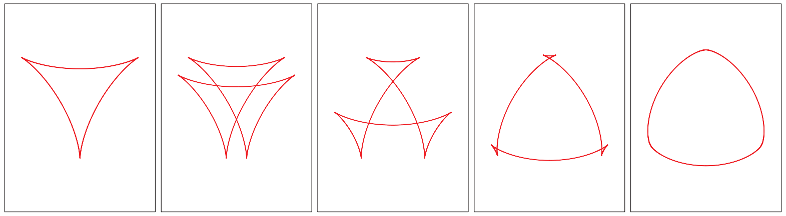
DEFINITION OF ARRANGEMENTS OF DOUBLE PSEUDOLINES



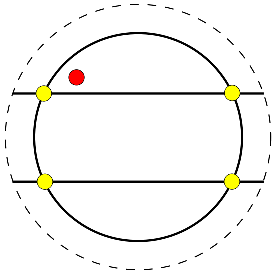
Definition 12. *Let $\mathcal{M}$ be an open Möbius band. A double pseudoline in $\mathcal{M}$ is the range of a simple loop in $\mathcal{M}$ homotopic to the double of a generator of $\pi_1(\mathcal{M})$, and an arrangement of double pseudolines in $\mathcal{M}$ is a finite family of double pseudolines in $\mathcal{M}$ that cross pairwise in exactly 4 points (the crossing being transversal) and that induce pairwise a cell structure on the one-point compactification of $\mathcal{M}$.* □

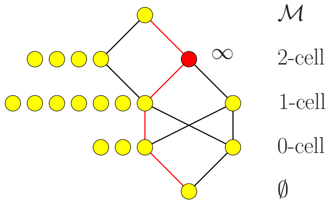
Lemma 21. *The dual of a planar family of pairwise disjoint convex bodies is an arrangement of double pseudolines.* □

DUAL OF A ϕ-CURVE



CHIROTOPES OF ARRANGEMENTS OF DOUBLE PSEUDOLINES





Definition 13. *Let $\mathcal{M}$ be an open Möbius band. Two arrangements defined in $\mathcal{M}$ are termed* <u>*isomorphic*</u> *if there is some homeomorphism of $\mathcal{M}$ to itself that maps the double pseudolines in one arrangement to those in the other. Two arrangements are termed* <u>*isotopic*</u> *if there exists a homotopy of isomorphic arrangements connecting them.* □

Definition 14. *The* <u>*chirotope*</u> *of an arrangement is the map that assigns to each subset of indices of size at most three the isotopy class of the subarrangement indexed by this subset (note that two isotopic arrangements have the same chirotope).* □

EXTENSION OF THE RINGEL THEOREM

Theorem 22. *Any pair of simple arrangements, defined on the same set of indices, are homotopic via a finite sequence of* <u>*mutations*</u> *followed by an isotopy.*

Theorem 23. *The map that assigns to an isotopy class of arrangements its chirotope is one-to-one.* □

Theorem 24. *Let I be a finite set of indices and let $\mathcal{C}_k(I)$, $k \in \mathbb{N}$, be the set of maps that assign to each triple of I an isotopy class of arrangements of double pseudolines indexed by that triple and whose restrictions to triples of subsets of indices of size at most k are chirotopes of arrangements of double pseudolines. Then*

$$\mathcal{C}_5(I) = \mathcal{C}_6(I) = \cdots = \mathcal{C}_\infty(I).$$

Theorem 25. *The map that assigns to a finite planar family of pairwise disjoint convex bodies its dual arrangement induces a one-to-one correspondance between the set of chirotopes of planar family of pairwise disjoint convex bodies and the set of chirotopes of arrangements of double pseudolines.* □

NOTES

ISO. CLASSES SIMPLE ARRANG. THREE DOUBLE PSEUDOLINES

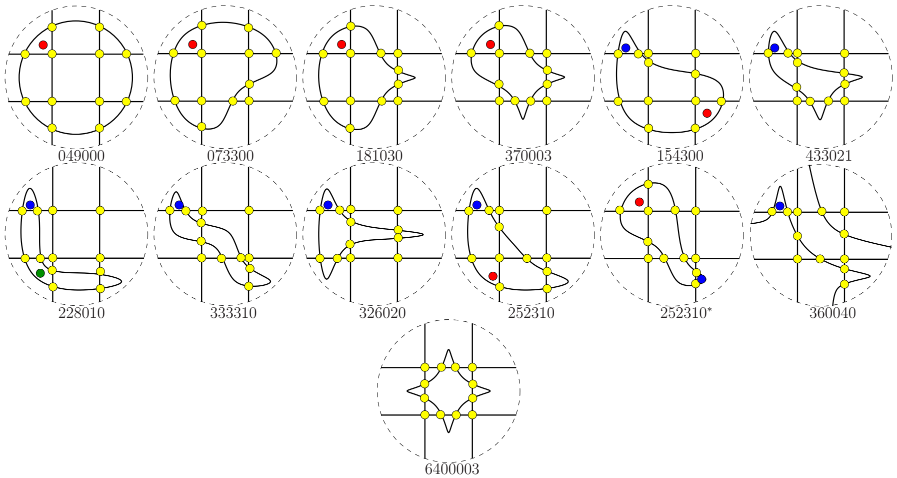
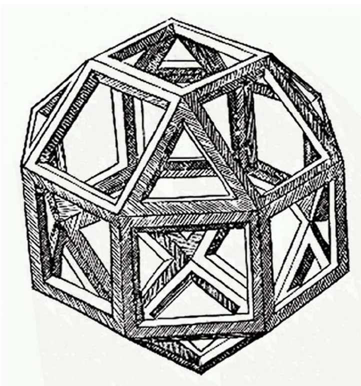


Figure 8: Arrangements of three double pseudolines in the projective plane; the last one is not realizable in the Möbius band because the union of the Möbius bands bounded by the double pseudolines covers the whole projective plane.

RHOMBICUBOCTAHEDRON



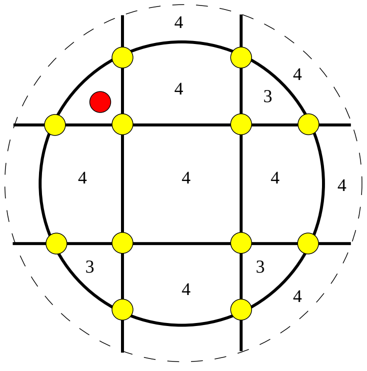


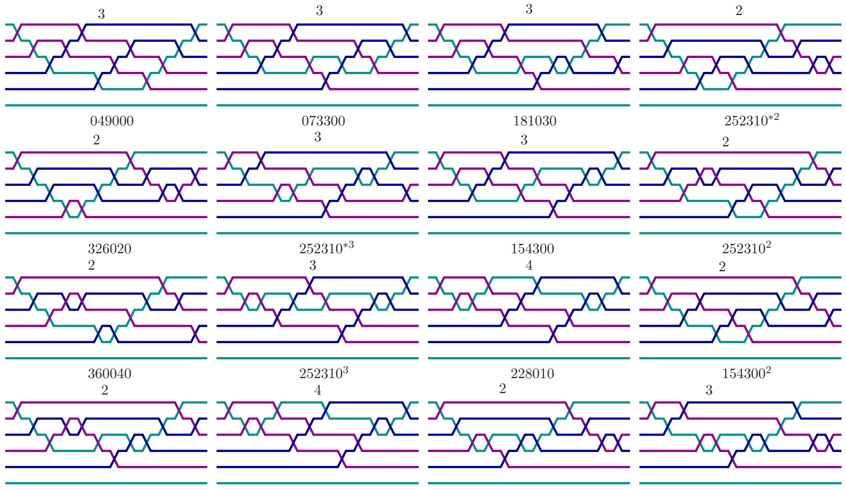
Figure 9: Première illustration d'un rhombicuboctaèdre en creux par Leonardo da Vinci pour la Divina proportione

COUNTING RESULTS

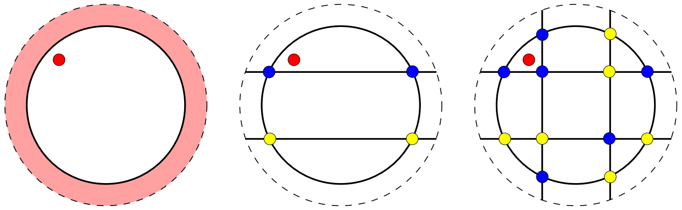
Preliminary counting results for simple arrangements are

n	2	3	4	5
a_n	1	118	541820	?
b_n	1	22	22620	?
c_n	1	16	11502	?
d_n	1	12	5955	?

An independent implementation by the Mezzenile research group confirms these values.

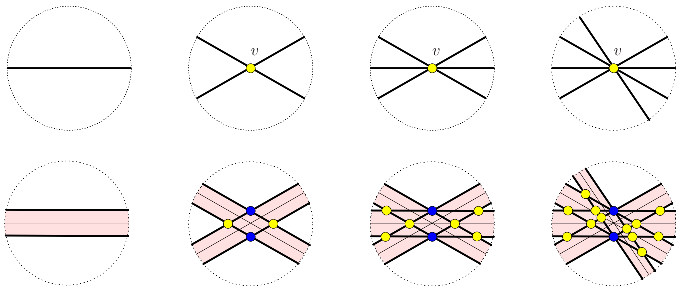


THIN ARRANGEMENTS



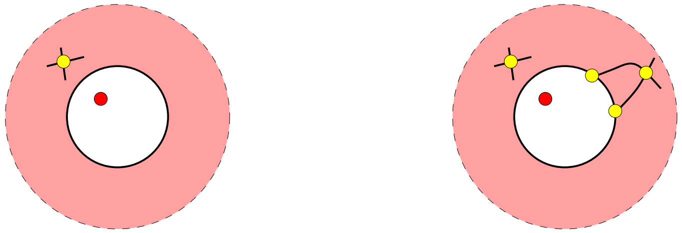
An arrangement is termed it thin if the Móbius bands of its double pseudolines are free of exterior vertices.

THICKENING



Theorem 26. *The thickening transformation induces a one-to-one and onto correspondence between the set of isotopy classes of pseudolines arrangements and the set of isotopy classes of thin double pseudolines arrangements, defined on the same set of indices.* □

PROOF OF THEOREM 22



Essentially based on the following lemma

Theorem 27 (Pumping Theorem)**.** *Let Γ be a simple double pseudoline arrang., X the induced cell structure of the open Möbius band $\mathcal{M}$, and $\gamma \in \Gamma$. Assume that there is a vertex of X lying in the interior of the Möbius band M bounded by γ. Then there is a triangular face of X included in M with a side supported by γ.* □

We split the proof into several lemmas. We work in a 2-covering $p : \tilde{\mathcal{M}} \to \mathcal{M}$ of $\mathcal{M}$; two curves of $\tilde{\Gamma}$ have exactly 0 or 2 intersection points depending on whether their are the lifts of the same curve in Γ, or not; we denote by $B \mapsto B_*$ the map that assigns to an intersection point between two curves of $\tilde{\Gamma}$ their second intersection point. the 2 lifts of γ under p are denoted γ^+ and γ^-; we choose an orientation for γ^+. Given a sequence of points $x, y, \ldots z$ on γ^+ we say that $xy \ldots z$ holds if walking along γ^+ we encounter the points $x, y \ldots, z$ in cyclic order $xy \cdots zxy \cdots$.

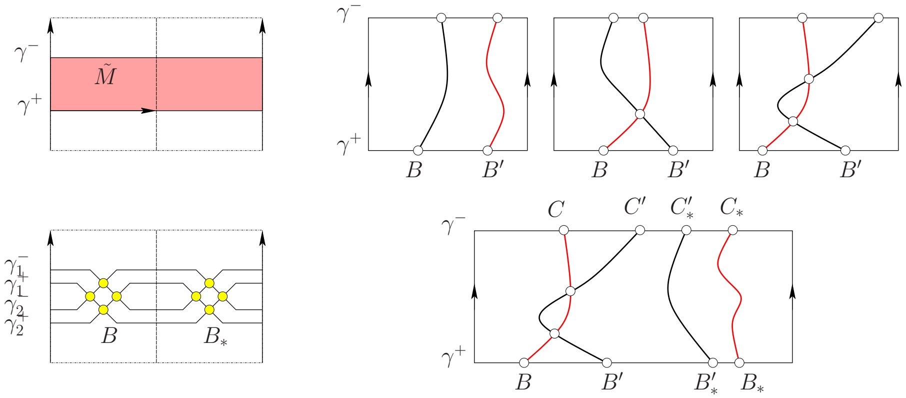


Figure 10: γ-curve story.

Define a γ-curve as a maximal subcurve of $\tilde{\gamma}' \in \tilde{\Gamma}$, $\gamma' \neq \gamma$, with range included in $\tilde{M}$; there are two γ-curves per $\tilde{\gamma}' \in \tilde{\Gamma}$, $\gamma' \neq \gamma$, and a γ-curve has an endpoint on γ^+ and the other one on γ^-; we orient the γ-curves from γ^+ towards γ^-; we denote by $\tau(B)$ the γ-curve with origin B.

Lemma 28. *Let $\tau(B)$ and $\tau(B')$ be two γ-curves crossing twice, let A be their first intersection point, and assume that the subcurve BB' of γ^+ is an edge of the triangle $BB'A$ of $\tilde{M}$. Then $BB'B'_*B_*$ holds.* □

ADMISSIBLE TRIANGLE AND ITS DERIVED ONE

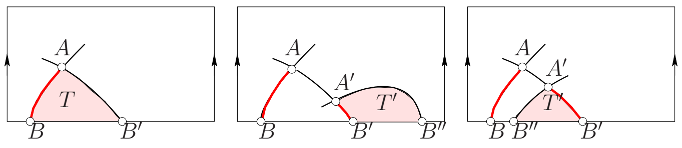


Figure 11: An admissible triangle T and its derived admissible triangle T'.

We define a *triangle* as a triangular 2-cell of the cell decomposition of $\tilde{M}$ induced by two crossing γ-curves with a side supported by γ^+; the vertex of a triangle not on γ^+ is called its *apex*. A triangle is said to be *admissible* if one of its two sides with endpoint its apex is a 1-cell of $\tilde{X}$.

Let $T = ABB'$ be an admissible triangle with apex A and 1-cell side AB, and let A' be the vertex of $\tilde{X}$ that follows B' on the side AB' of T. Then A' is the apex of an admissible triangle $T' = A'B'B''$ with 1-cell side $A'B'$.

Lemma 29. *The sequence $T, T', T'', \ldots$ is stationnary.*

Proof. Assume wlog that walking along γ^+ we traverse the side BB' of T from B to B'. We distinguish two cases depending on whether $BB'B''$ holds or not. SEE NEXT TWO SLIDES. □

TWO CASES

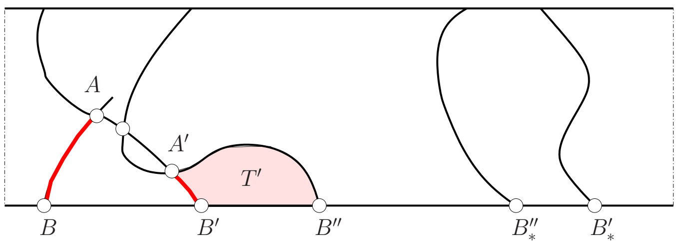


Figure 12: Case 1.

Case 1. Assume that $BB'B''$ holds. Then $\tau(B')$ and $\tau(B'')$ cross twice and $BB'B''B''_*B'_*$ holds.

Case 2. Assume that $B''B'B, B'''B''B', B^{(4)}B'''B'', \ldots, B^{(k+1)}B^{(k)}B^{(k-1)}$ hold for some $k \geq 1$. Then $\tau(B^{(j+1)})$ and $\tau(B^{(j)})$, $2 \leq j < k$, cross twice and $BB^{(k+1)}B^{(k)} \ldots B''B'$ holds.

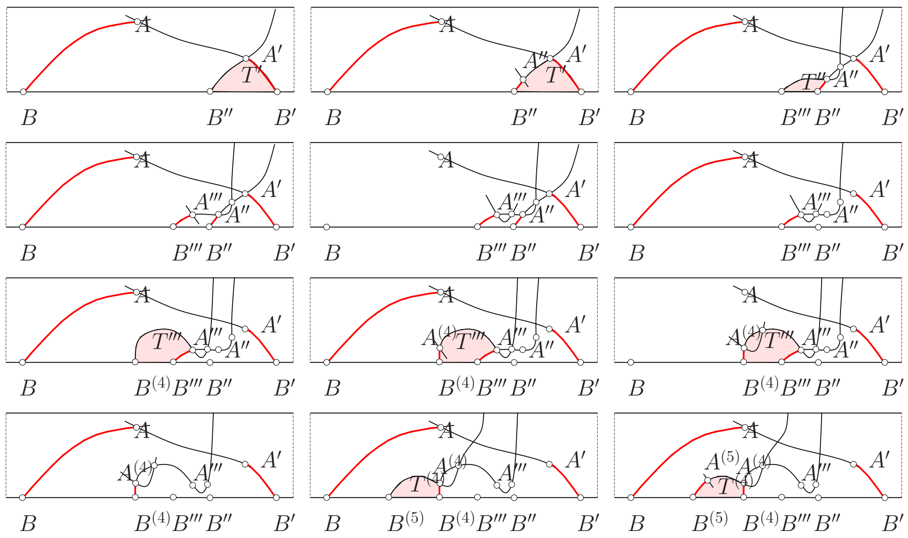


Figure 13: Case 2.

SHORT BIBLIOGRAPHY

F. Levi (1926), G. Ringel (1956), B. Grünbaum (1972), J. E. Goodmann (1980), R. Cordovil (1982), M.E. Mnëv (1988), P. Shor (1991), J.E. Goodmann & R. Pollack & R. Wenger and T. Zamfirescu (1994), . . . , Edelsbrunner & Guibas (1989),. . . , Agarwal & Sharir (2005).

References

[1] L. Habert and M. Pocchiola. Arrangements of double pseudolines in Möbius strips. In preparation, 2006.

[2] L. Habert and M. Pocchiola. An homotopy theorem for arrangements of double pseudolines. In Abstracts 12th European Workshop Comput. Geom., pages 211–214, 2006.

[3] Jacob E. Goodman. Pseudoline arrangements. In Jacob E. Goodman and Joseph O'Rourke, editors, Handbook of Discrete and Computational Geometry, chapter 5, pages 97–128. Chapman & Hall/CRC, 2004.

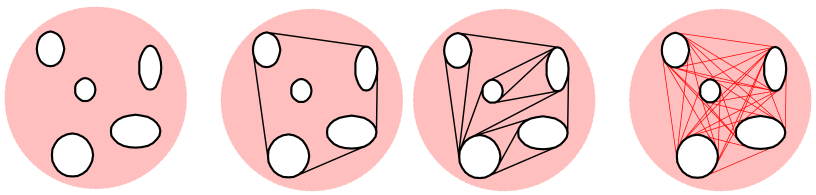
NOTES

NOTES

NOTES

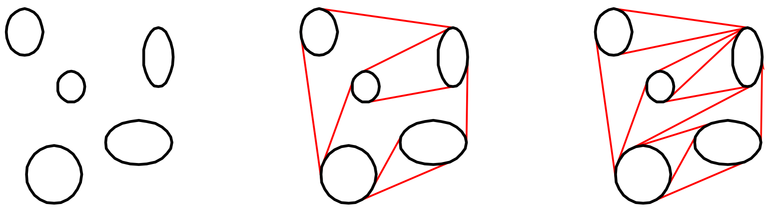
NOTES

PART II: ALGORITHMES



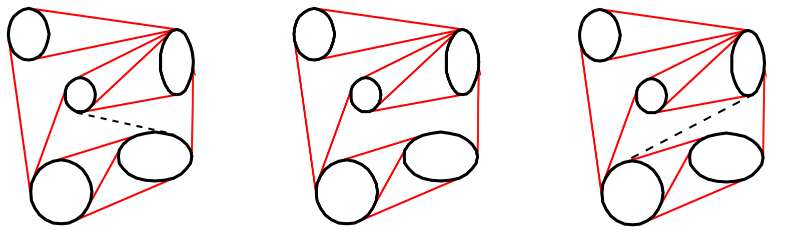
Theorem 30 (Habert & P. 04/06, Angelier & P. 03, P. & Vegter 96). *The convex hull, a (greedy) pseudotriangulation, the visibility graph of a finite family of pairwise disjoint convex bodies of a topological plane on $\mathbb{R}^2$ are each computable using only O(size of the output+ size of the input $\times$ $\log$ size of the input) probes of its chirotope.* □

PSEUDOTRIANGULATIONS



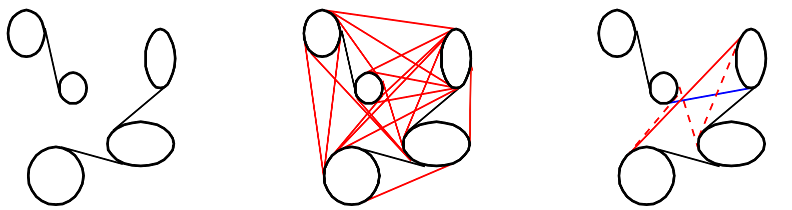
1. O a set of n pairwise disjoint convex bodies
2. B the set of $k \leq 2n(n-1)$ undirected free bitangents
3. A pseudotriangulation G is a maximal (for $\subseteq$) planar subset of B
4. $C \subset G$, $|G| = 3n - 3$
5. $(\forall G)\,(\forall b \in G \setminus C)\,(\exists!\, G' \neq G)\;\; (G \setminus G' = \{b\})$.

FLIP GRAPH OF PSEUDOTRIANGULATIONS



1. G a pseudotriangulation, $C \subset G$, $|G| = 3n - 3$
2. **[Flips]** $(\forall G)\,(\forall b \in G \setminus C)\,(\exists!\, G' \neq G)\;\; (G \setminus G' = \{b\})$ $\qquad G \xrightarrow{b} G'$.
3. **[Connectivity]** Le graphe d'adjacence des pseudotriangulations est 'fortement' connexe.
$$\begin{cases} G = G_0 \xrightarrow{b_1} G_1 \cdots \xrightarrow{b_k} G_k = G' \\ G \cap G' \subseteq G_i \qquad (i = 0, \ldots, k) \end{cases}$$
4. $\{H \mid C \subseteq H, H$ planaire$\}$ ordonné par inclusion est un polytope abstrait de dimension $3n - 3 - |C|$.

GREEDY PSEUDOTRIANGULATIONS



1. $\kappa \subset \mathcal{B}$ planar
2. $\mathcal{B}_\kappa = \{v \in \mathcal{B} \mid \kappa \cup \{v\}$ is planar$\}$, $\Theta(v)$, $\mathbb{B}_\kappa = \mathcal{B}_\kappa \times \mathbb{Z}$, $\Theta(v, k) = \Theta(v) + k\pi$
3. We define a partial order $\prec_\kappa$ on $\mathbb{B}_\kappa$ as follows : $v \prec_\kappa v'$ if there exists a sequence of bitangents $v = v_1, v_2, \ldots, v_k = v'$ such that v_i and v_{i+1} touch the same directed disk and $\Theta(v_i) < \Theta(v_{i+1})$

Theorem 31 (PV96,AP03)**.** *Two crossing bitangents of $\mathbb{B}_\kappa$ are comparable with respect to $\prec_\kappa$ and the operator $\varphi(.;\kappa) : \mathbb{B}_\kappa \to \mathbb{B}_\kappa$ that associates with $v \in \mathbb{B}_\kappa$ the minimum element of the set of $u \in \mathbb{B}_\kappa$ such that u crosses v and $v \prec_\kappa u$ is well-defined, one-to-one, and onto.* □

GREEDY FLIP PROPERTY

For any filter I of the poset $(\mathbb{B}_\kappa, \prec_\kappa)$, we define a maximal planar set of bitangents $\mathrm{G}(I) = \{v_1, v_2, \ldots, v_k\}$ recursively: v_1 is minimal in I, and, for $i > 1$, v_{i+1} is minimal in the set of bitangents $v \in I \setminus \{v_1, \ldots, v_i\}$ such that $\kappa \cup \{v_1, \ldots, v_i, v\}$ is planar—note that this set is empty after $3n - 3$ steps.

Theorem 32 (PV96,AP03)**.** *Let v be a minimal element of the filter I of the poset $(\mathbb{B}_\kappa, \prec_\kappa)$. Then*

1. $\mathrm{G}(I) = I \setminus \varphi(I; \kappa)$;
2. $\mathrm{G}(I \setminus \{v\})$ *is obtained from* $\mathrm{G}(I)$ *by flipping* v. □

ENUMERATION OF PSEUDOTRIANGULATIONS

Let I be a filter (for example the filter of bitangents with positive angle) of $(\mathbb{B}_\kappa, \prec_\kappa)$, and let $G_0 = G(I)$ be the associated greedy pseudotriangulation. For $H = (F, G)$, $F \cap G = \emptyset$, $\kappa \subset F \cup G \subset I$ a pseudotriangulation, we define $\omega(H)$ to be the pair $(F \cup \{v\}, G \setminus \{v\})$ and $\phi(H)$ to be the pair $(F, G \setminus \{v\} \cup \{v'\})$ where $v = \min_{<_\kappa} G$ and v' is the shadow of v in $F \cup G$. (Observe that ω and ϕ are defined only if $G \neq \emptyset$.) For $G \supseteq \kappa$ a pseudo-triangulation of the o_i's we define a sequence $H(G) = H_1, H_2, \ldots, H_i = (F_i, G_i)$ by setting $H_0 = (\emptyset, G_0)$, and

$$H_{i+1} = \begin{cases} \omega(H_i), & \text{if } \min_{<_\kappa} G_i \in G; \\ \phi(H_i), & \text{otherwise.} \end{cases} \tag{1}$$

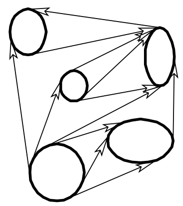
Il est aisé de voir that the sequence $H(G)$ is finite, that its last term is $\overline{H}(G) = (G, \emptyset)$, and that $H(G) = H(G')$ iff. $G = G'$. This proves the strong connectivity of the flip graph of pseudotriangulations and, with some extra work, provides a linear space traversal algorithm of the flip graph whose complexity is linear amortized per node

❏ — Brönnimann et al'01, Bespamyatnikh'02

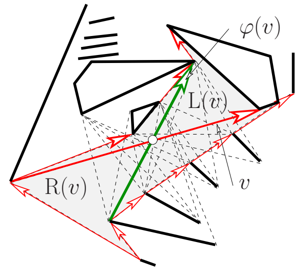
VISIBILITY GRAPH ALGORITHM

Algorithm Greedy Flip Algorithm

compute the greedy pseudo-triangulation $G := \mathrm{G}(I(0))$;
repeat
  select a minimal bitangent v in G with angle less than π;
  flip v; (i.e., replace v by $\varphi(v)$)
until there are no more bitangents of angle less than π.



$\mathrm{G}(S)$



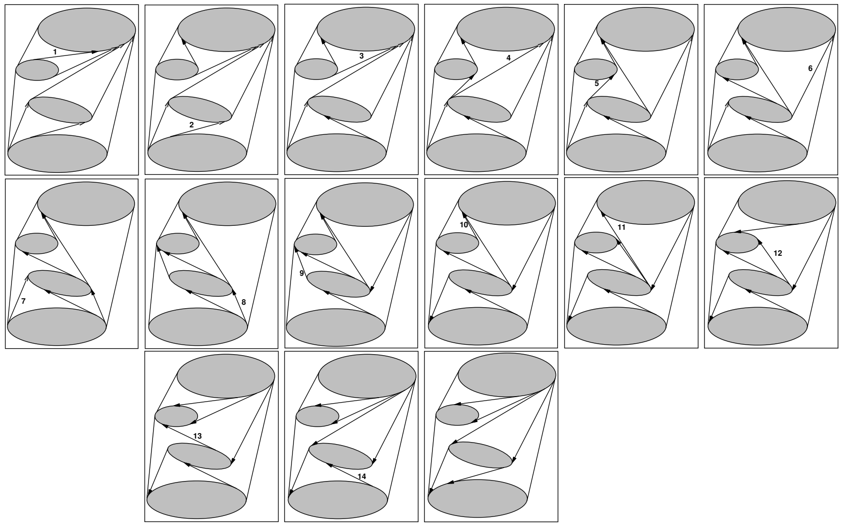
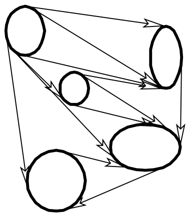


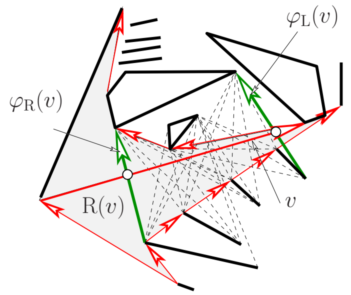
Figure 14: The greedy flip algorithm. At each step the internal bitangent of minimal slope in the current pseudo-triangulation is flipped

THE FLIP OPERATION (VERSION AP03)

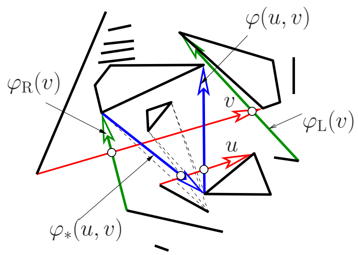
Theorem 33 (AP03)**.** *The flip operation of the greedy flip algorithm can be implemented in constant amortized time* <u>*using only the chirotope of the convex bodies.*</u> *Furthermore the only necessary data structures are linked structures to represent the incidence relations in the sweep structure S and its associated pseudotriangulations $\mathrm{G}(S)$ and $\mathrm{G}_*(S)$, a list to store the minimal bitangents of $\mathrm{G}(S)$, and for each minimal bitangent v pointers to the arcs of $\mathrm{G}(S)$ that $\varphi_{\mathrm{R}}(v)$ and $\varphi_{\mathrm{L}}(v)$ leave and pointers to the arcs of $\mathrm{G}_*(S)$ that $\varphi_{\mathrm{R}}(v)$ and $\varphi_{\mathrm{L}}(v)$ enter.* □

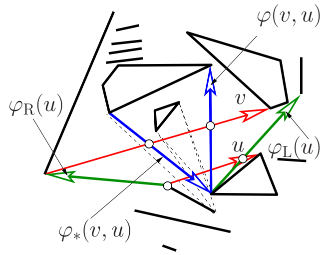


$\mathrm{G}_*(S)$



"SUM OF SQUARES" THEOREM

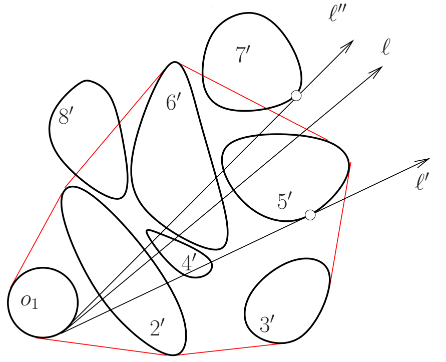




$$\varphi(u, v) = \varphi(u; \{\varphi_{\mathrm{R}}(v), \varphi_{\mathrm{L}}(v)\}).$$

Theorem 34 (AP03)**.** *The number of pairs of free bitangents (u, v) such that $\varphi(u, v) = \varphi(v, u)$ and $\varphi_*(u, v) = \varphi_*(v, u)$ is bounded by a constant times the number of free bitangents.* □

INITIALIZATION OF THE GFA



Theorem 35 (HP03)**.** *There is a $\mathrm{G}(I)$ computable in $O(n \log n)$ time using only the chirotope of the convex bodies.* □

BIBLIOGRAPHIE

[1] L. Habert et M. Pocchiola. – Computing pseudotriangulations via branched coverings. – En préparation, 2006.

[2] P. Angelier and M. Pocchiola. A sum of squares theorem for visibility complexes and applications. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, <u>Discrete and Computational Geometry – The Goodman-Pollack Festschrift</u>, Algorithms and Combinatorics, volume 25, pages 79–139. Springer-Verlag, June 2003.

[3] P. Angelier and M. Pocchiola. CGAL-Based implementation of visibility complexes <u>Technical Report ECG-TR-241207-01</u>, École Normale Supérieure, Paris, France 2003.

[4] P. Angelier. <u>Algorithmique des graphes de visibilité</u>. PhD thesis, École Normale Supérieure (Paris), February 2002.

[5] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. <u>Discrete Comput. Geom.</u>, 16(4):419–453, December 1996.

[6] M. Pocchiola and G. Vegter. The visibility complex. <u>Internat. J. Comput. Geom. Appl.</u>, 6(3):279–308, 1996.

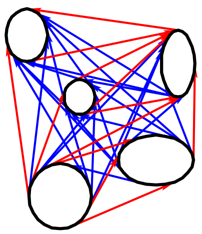
NOTES

NOTES

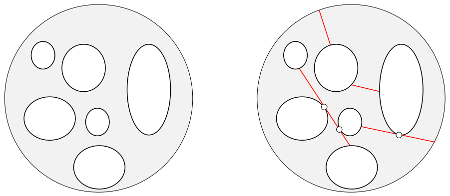
NOTES

NOTES

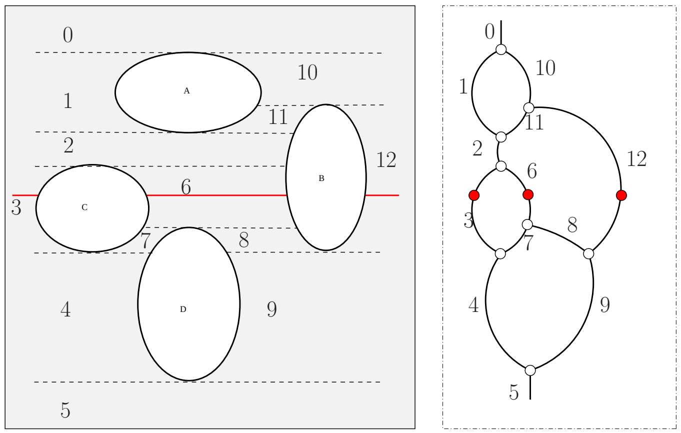
COMPLEXES DE VISIBILITÉ



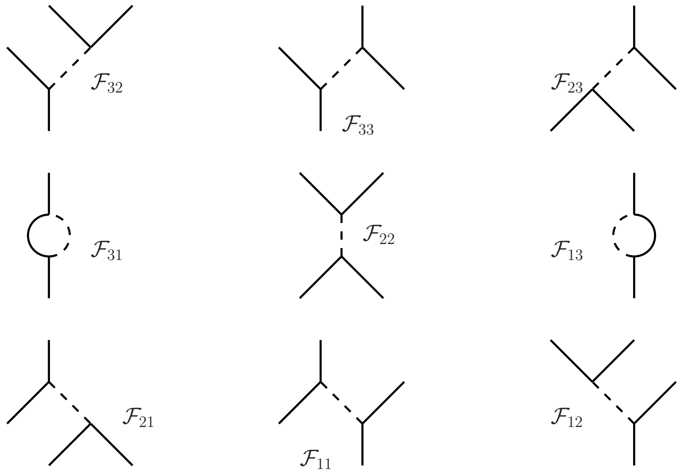
1. UNE COLLECTION DE n CORPS CONVEXES : $O_i \subset \mathbb{R}^2$
2. ESPACE LIBRE : $\mathbb{F} = \mathbb{R}^2 \setminus \bigcup_i \text{INTERIOR}(O_i)$
3. ESPACE DES DROITES DE $\mathbb{F}$: $\mathbb{V}$
4.



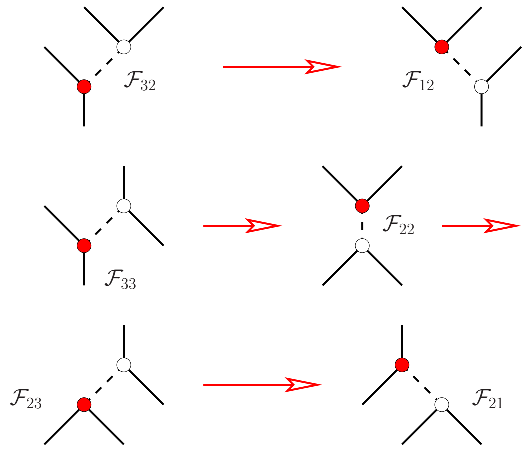
CROSS-SECTIONS OF THE VISIBILITY COMPLEX



TYPOLOGY OF THE 2-FACES OF A CROSS-SECTION



TRANSITIONS : $\mathcal{F}_{i+1,j+1} \to \mathcal{F}_{j,i}$



BALAYAGE DROIT

1. Fonctions supports des $\{o_i\}$

$$\theta \in \mathbb{R} \;\mapsto\; z_{+o}(\theta) = \inf_{p\in o}\langle p, \imath\exp(\imath\theta)\rangle \in \mathbb{R}$$
$$\theta \in \mathbb{R} \;\mapsto\; z_{-o}(\theta) = \sup_{p\in o}\langle p, \imath\exp(\imath\theta)\rangle \in \mathbb{R}$$

2. $z_{-o}(\theta+\pi) = -z_{+o}(\theta)$

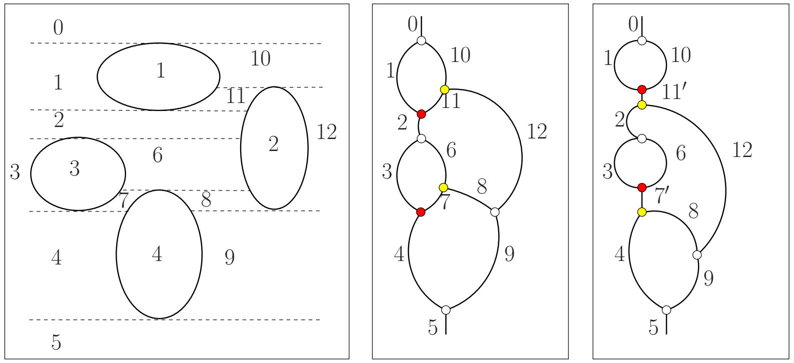
Theorem 36. *Soit $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k < \pi$ la suite croissante des angles des bitangentes libres orientées vers le haut et pour toute arête σ de la section $S_i \approx S(\theta)$ pour $\theta \in (\theta_{i-1}, \theta_i)$ du complexe de visibilite soit* $\mathrm{death}(\sigma)$ *la plus petite solution supérieure à θ_{i-1} de l'équation $z_{\sigma^+}(\theta) = z_{\sigma^-}(\theta)$. Alors θ_i est le plus petit des* $\mathrm{death}(\sigma)$ *pour σ arête de S_i.* □

Theorem 37. *Les k bitangentes libres d'une collection de n corps convexes disjoint deux à deux d'un plan topologique affine sont calculables en temps $O(k\log n)$ et espace de travail linéaire.* □

EXAMPLE

$2 \in \mathcal{F}_{22}$, $9 \in \mathcal{F}_{32}$, $7, 11 \in \mathcal{F}_{33}$, $12 \in \mathcal{F}_{31}$, $1, 3 \in \mathcal{F}_{13}$, $6, 10 \in \mathcal{F}_{21}$, $8 \in \mathcal{F}_{11}$, $4 \in \mathcal{F}_{12}$, $5 = \hat{0}$, $0 = \hat{1}$. $\mathrm{death}(2) = v_{-3,1}$, $\mathrm{death}(11) = v_{1,-2}$, $\mathrm{death}(9) = v_{4,2}$, $\mathrm{death}(7) = v_{3,-4}$.

$$\mathrm{death}(7) = \mathrm{death}(11) < \mathrm{death}(2) < \mathrm{death}(9)$$



BORDS GAUCHE ET DROIT D'UNE 2-CELLULE

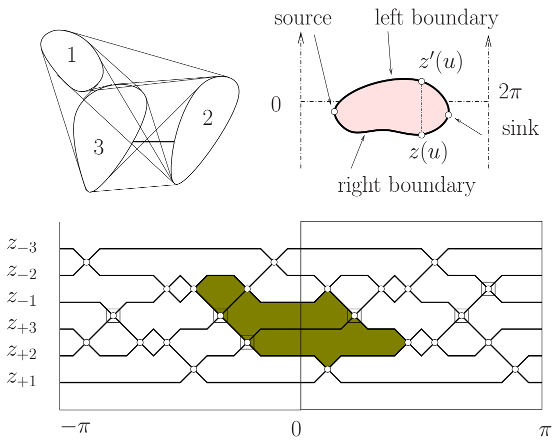
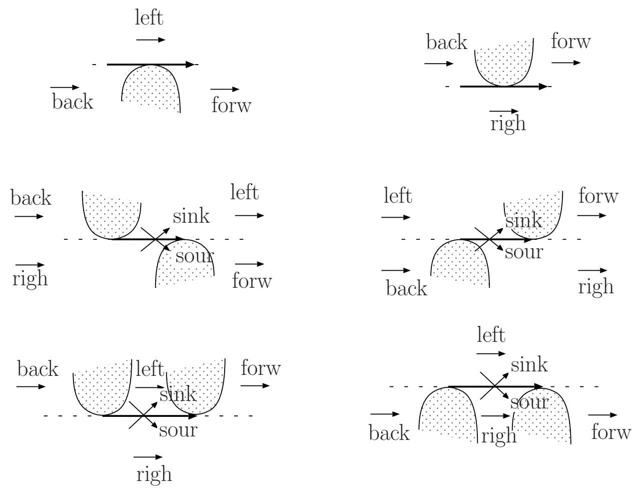


Figure 15: Example of arrangement of curves $z_{\epsilon o}$'s for a collection of three convex bodies : each curve $z_{\epsilon o}$ is stretched and bended so that it stays roughly parallel to the equator of the space of directed lines. There is a one-to-one and onto correspondence between the set of vertices of the arrangement and the set of free and non-free directed bitangents. The (6) vertices marked with a square correspond to non-free bitangents. The dashed region represents the projection in the space of directed lines of the 2-cell containing the elements of the visibility complex with backward view the disk labeled 3 and forward view the disk labeled 2.

INCIDENCE RELATIONS



INCIDENCE OPERATORS

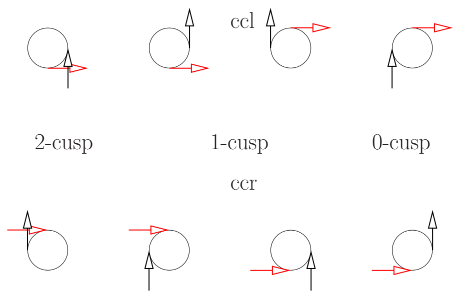


Figure 16: Regular, cusp and double cusp arcs.

1. Sink and source of a 1- or 2-cell;
2. ccr and ccl operators;
3. For $\sigma \in X_0 \cup X_1$ and $\alpha \in \{\text{left}, \text{back}, \text{righ}, \text{forw}, \text{sink}\}$ we set (if defined)

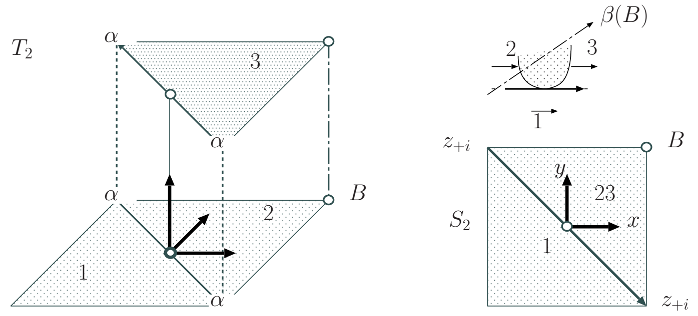
$$\varphi_\alpha(\sigma) = \text{sink}(\sigma_\alpha), \qquad \alpha \in \{\text{left}, \text{back}, \text{righ}, \text{forw}, \text{sink}\}. \tag{2}$$

COMPUTING THE RIGHT BOUNDARY OF A FACE

Here is an algorithm to compute the right boundary of the face σ whose source is b.
TO BE CHECKED

1 push(b); $b_2 \leftarrow$ ccr(b);
2 **if** (b, b_2) 2-cusp **then goto 6**
3 **while**(b, b_2) 0-cusp {
push(b_2); /* b_2 IS 'FAT' */
$b_2 \leftarrow$ ccl(b_2);
}
4 **do** {
push (b_2); /*b_2 IS 'REGULAR' */
$(b_1, b_2) \leftarrow (b_2, \text{ccl}(b_2))$;
} **while** (b_1, b_2) 0-cusp
5 **while** (b_1, b_2) 2-cusp {
push (b_2); /* b_2 IS 'FAT'*/
$b_2 \leftarrow$ ccr(b_2);
}
6 push(b_2); /*b_2 IS THE SINK OF σ, I.E., $b_2 = \varphi_{\text{sink}}(b)$ */
exit;

TOPOLOGY OF THE VISIBILITY COMPLEX



TOPOLOGY OF THE VISIBILITY COMPLEX