Network topology and the eficiency of equilibrium

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Networks

Igal Milchtaich, Games and Economic Behavior 57 (2006), 321-346

$G = (V, E)$ a finite undirected graph.

s, t a couple of source - sink (destination) vertices (single commodity network).

Each edge $e \in E$ joins two distinct vertices $v, u \in V$ (loops are not allowed but multiple parallel edges are possible).

A path is a sequences

$$p = v_0 e_1 v_1 \ldots v_{n-1} e_n v_n$$

alternating vertices and edges, beginning and ending with vertices and such that each edge e_i is incident with vertices v_{i-1} and v_i and all vertices v_i are different.

A route is a path starting in the source vertex s and ending in the sink vertex t.

$\mathcal{R}$ is the set of all routes in the network.

$\mathcal{P}$ is the set of all paths.

For a path p going through vertices u and w, where u precedes w on p,

$$p_{uw}$$

is the section of p starting at u and ending at w.

Flows

$f : \mathcal{R} \longrightarrow \mathbb{R}^+$ the flow mapping.

f_r denotes the flow through a route $r \in \mathcal{R}$.

The total flow or traffic is defined as

$$|f| = \sum_{r \in \mathcal{R}} f_r$$

If p is a path then then flow through p is defined as

$$f_p = \sum_{\substack{r \in \mathcal{R} \\ p \text{ is a section of } r}} f_r$$

An arc of a network G is a path of the form uev, i.e. a path containing one edge. (An arc $=$ oriented edge).

Cost function

Let f a flow and p a path.

$$c_p(f)$$

the cost of the path p as a function of the (entire) flow f.

We assume that c is monotone in the following sense:

for every path p and all flows f and f^*, if $f_p \geq f_p^*$ and $f_{-p} \geq f_{-p}^*$, where $-p$ is the path inverse to p, then $c_p(f) \geq c_p(f^*)$.

This implies, in particular, that the cost of each path depends only on the flow on each of its arcs in the direction of p and in the opposite direction.

The cost function is increasing if it satisfies the following condition:

for each route r and all flows f and f^* if $f_p \geq f^*(p)$ and $f_{-p} \geq f^*_{-p}$ for each arc p of r then $c_r(f) \geq c_r(f^*)$.

A cost function is additive if for each path p and each flow f

$$c_{p_{uv}}(f) = c_{p_{uw}}(f) + c_{p_{wv}}(f)$$

where u, w, v are vertices appearing in this order when we traverse p.

Additivity means that the cost of a path is the sum of the costs of its arcs.

Equilibrium

A flow f is in (Nash/Wardrop) equilibrium if the entire flow is on on routes of minimal cost, that is,

for all routes $r \in \mathcal{R}$ with $f_r > 0$,

$$c_r(f) = \min_{q \in \mathcal{R}} c_q(f)$$

For an equilibrium f we denote

$$c(f) = \min_{q \in \mathcal{R}} c_q(f)$$

the equilibrium cost (for each user).

If the cost functions are continuous then for any traffic $a > 0$ there exists and equilibrium flow f such that $|f| = a$.

Braess's paradox

Braess's paradox occurs in a network G if there exist two additive cost functions c and c^* such that for all routes $r \in \mathcal{R}$ and all flows $f^\sharp$, $c_r(f^\sharp) \geq c^*_r(f^\sharp)$ but for every equilibrium flow f with respect to c and every equilibrium flow f^* with respect to c^* such that $|f| = |f^*|$ the equilibrium costs satisfy

$$C(f) < C^*(f^*)$$

Theorem 1. *Braess's paradox does not occur in a network G iff and only if G is series-parallel.*

Series-parallel networks

A network G is series-parallel if either it contains only one edge or it is obtained by a composition in parallel or in series of two series-parallel networks.

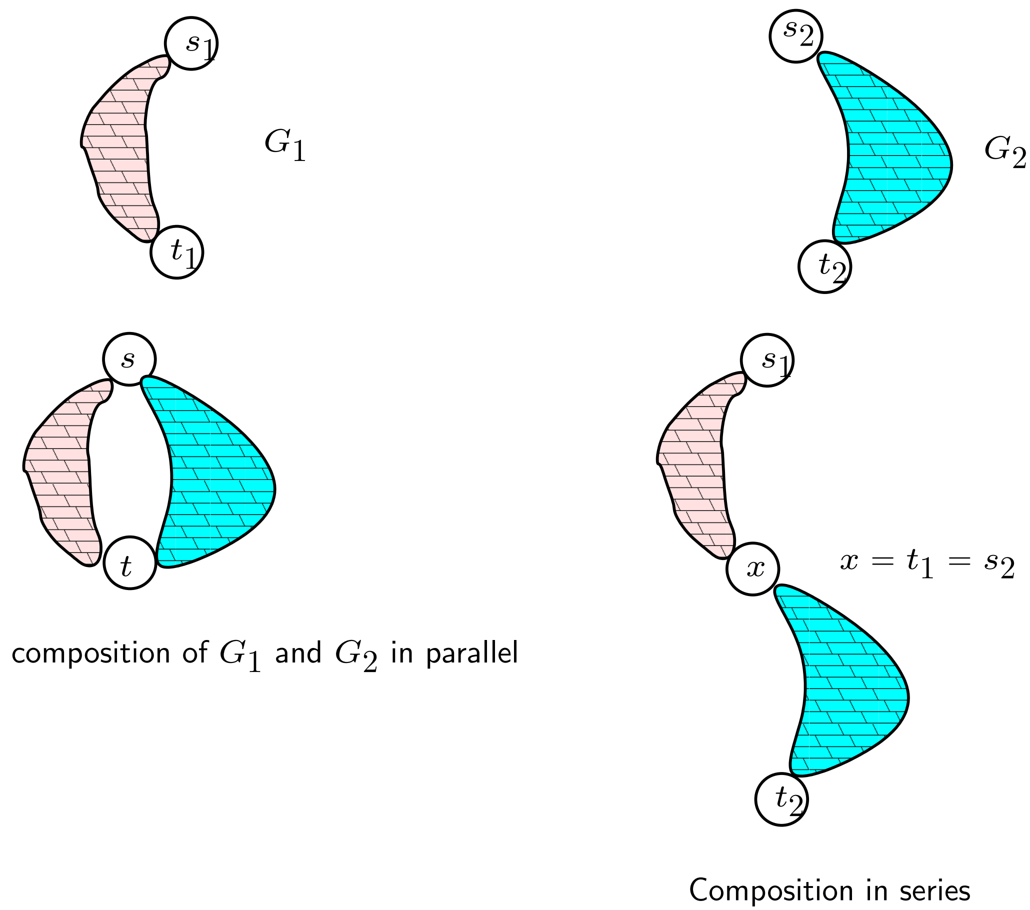


Figure 1: Compositions in parallel and in series of two networks

Wheatstone network

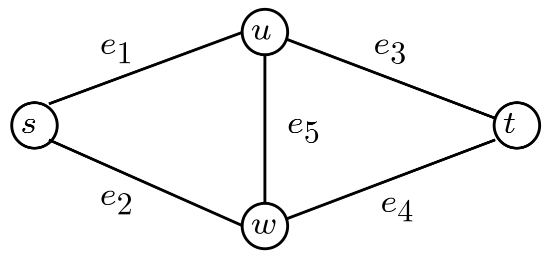


Figure 2: Wheatstone network

$c_{e_1}(x) = c_{e_4}(x) = 1 + 6x$, $c_{e_2}(x) = c_{e_3}(x) = 15 + 2x$, $c_{e_5}(x) = 1 + 6x$.

Total traffic 1.

In equilibrium all the flow goes through $e_1 e_5 e_4$ with the cost $3(1+6\cdot 1) = 21$.

Increase the cost of the edge e_5 to $c^*_{e_5}(x) = 15 + 2x$. Then in equilibrium half of the traffic goes through e_1e_3 and the other half through e_2e_4 with the cost for each user $(1 + 6 \cdot \frac{1}{2}) + (15 + 2 \cdot \frac{1}{2}) = 20$.

Embedding a network

A network G_1 is embedded in a network G_2 if either both networks are isomorphic or G_2 can be obtained from G_1 by applying any sequence of the following operations:

(1) the subdivision of an edge with a new vertex that divides the edge on two adjacent edges,

(2) an addition of an edge joining two existing vertices,

(3) a series composition with the one-edge network.

Proposition 1. *For a network G the following conditions are equivalent:*

(1) G is series-parallel,

(2) two routes never pass through any edge in opposite directions,

(3) for every pair of distinct vertices u and v, if u precedes v in some route r containing both vertices then u precedes v in all such routes.

(4) the Wheatstone network of Fig. 2 cannot be embedded in G,

(5) the vertices of G can be indexed by integers in such a way that, along each route, the indices are increasing.

Lemma 1. *Let G be series parallel and f, f^* two flows in G such that $|f| \geq |f^*| > 0$. Then there exists a route r such that for each arc p of r,*

$$f_p \geq f^*_p \quad \textit{and} \quad f_p > 0$$

If $|f| > |f^*| > 0$ then both inequalities above are strict.

Proof. Induction on the number of vertices of G, using compositions in series and in parallel. $\square$

Lemma 2. *Let G a series-parallel network, c an additive cost function, and f an equilibrium flow for c. Then for every route the following conditions are equivalent:*

(i) the route r is a minimal cost route (i.e. the cost of this route is equal to the cost of the flow f for the users, $C(f) = c_r(f)$),

(ii) every edge of r is in some minimal cost route.

Proof. (i)$\Longrightarrow$(ii) Obvious.

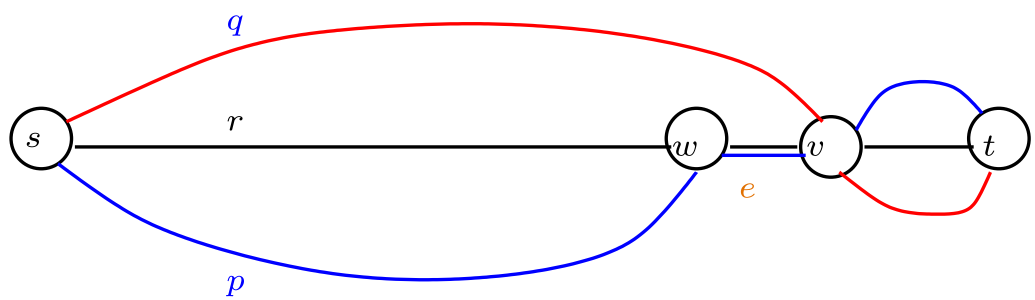
(ii)$\Longrightarrow$(i) Let r a route satisfying (ii).

We shall prove the following (stronger) claim.

Claim For every minimal-cost route q and every vertex v common to r and q, $c_{r_{sv}}(f) = c_{q_{sv}}(f)$.

By induction on the length of r_{sv}. Case $v = s$ trivial.

Suppose $s \neq v$ and w precedes v on r and e is the corresponding edge.



By (ii) there exists a minimal cost route p containing the edge e. By induction hypothesis

$$c_{r_{su}}(f) = c_{p_{su}}(f)$$

c additive thus

$$c_{r_{sv}}(f) = c_{r_{su}}(f) + c_e(f) = c_{p_{su}}(f) + c_e(f) = c_{p_{su}}(f)$$

To complete the proof of the claim it remains to show that for every pair of minimal cost routes p and q with a common vertex v

$$c_{p_{sv}}(f) = c_{q_{sv}}(f)$$

From Proposition 1 (5) it follows that $p_{sv}q_{vt}$ is a route in G. Thus, since q is cost minimal under f

$$c_{p_{sv}}(f) + c_{q_{vt}}(f) = c_{p_{sv}q_{vt}}(f) \geq c_q(f) = c_{q_{sv}}(f) + c_{q_{vt}}(f)$$

implying

$$c_{p_{sv}}(f) \geq c_{q_{sv}}(f)$$

The inverse inequality $c_{q_{sv}}(f) \geq c_{p_{sv}}(f)$ follows by symmetry.

□

Lemma 3. *Let G a series-parallel network, $c^\sharp$ and c^* cost functions such that for each flow f, $c^\sharp_r(f) \geq c^*_r(f)$ for all routes r. Let $f^\sharp$ and f^* equilibria for $(G, c^\sharp)$ and (G, c^*) respectively such that $|f^\sharp| \geq |f^*|$. Then the equilibrium costs satisfy $C^\sharp(f) \geq C^*(f)$ (i.e. Breass paradox does not occur in the network G).*

Proof. If $f^\sharp = f^*$ nothing to prove.

Suppose that $f^\sharp \neq f^*$, thus, in particular, $f^\sharp \neq 0$.

By Lemma 1 there exists a route r such that for each arc p of r,

$$f^\sharp_p \geq f^*_p \quad \text{and} \quad f^\sharp_p \neq 0.$$

Since $c^\sharp$ is additive the first inequality implies that $c^\sharp_r(f^\sharp) \geq c^\sharp_r(f^*)$ while the second implies that every edge p of is on some route q with $f^\sharp_q > 0$. Te equilibrium condition and Lemma 2 imply that

$$c^\sharp_r(f^\sharp) = C^\sharp(f^\sharp)$$

Hence $C^\sharp(f^\sharp) \geq c^\sharp_r(f^*)$.

But by our assumption

$$c_r^\sharp(f^*) \geq c_r^*(f^*)$$

and by definition of equilibrium

$$c_r^*(f^*) \geq C^*(f^*)$$

Thus

$$C^\sharp(f^\sharp) \geq C^*(f^*)$$

□

The preceding lemma shows that Braess's paradox does not occur in series-parallel graphs. In series-parallel graphs Wheatstone network can be embedded and we can mimic the construction of the Braess's paradox for this network.

Pareto efficiency

Let G a network, c a cost function and f^* an equilibrium flow for (G, c). Then f^* is weakly Pareto efficient if, for every flow f such that $|f^*| = |f|$ there is some route r with $f_r > 0$ such that $c_r(f) \geq C(f^*)$.

The equilibrium f^* is Pareto efficient if, for every flow f such that $|f| = |f^*|$, either $c_r(f) = C(f^*)$ for all routes r with $f_r > 0$ or there is some route r with $f_r > 0$ for which $c_r(f) > C(f^*)$.

Pareto inefficiency

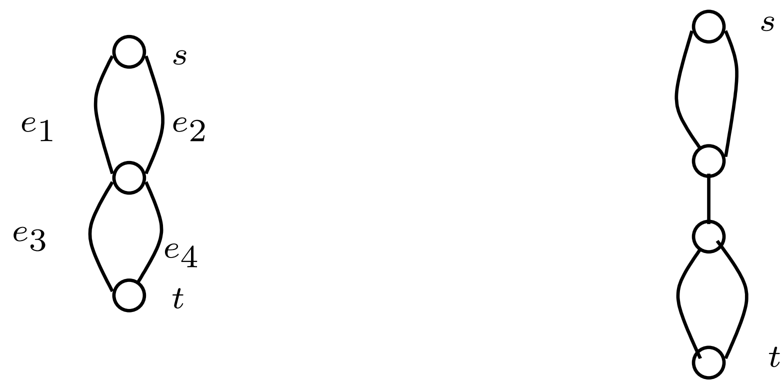


Figure 3: Pareto inefficient networks

For e_1 and e_3 the cost is $2x$, for e_2 and e_4 is is $2 + x$. The total flow is 1.

At equilibrium only e_1 and e_3 are used and the user cost is 4. But splitting the flow, half through e_1e_4 half through e_2e_3 the user cost is $3, 5$. The equilibrium is not Pareto efficient.

Theorem 2. *For a network G the following conditions are equivalent:*

1. *for any cost function, all equilibria are weakly Pareto efficient,*

2. *for any increasing cost function, all equilibria are Pareto efficient,*

3. *G has linearly independent routes.*

A network G has linearly independent routes if each route has at least one edge that does not belong to any other route.

This is a subclass of series-parallel networks.

Proposition 2. *For each network G the following conditions are equivalent:*

(i) G has linearly-independent routes,

(ii) for every pair of routes r and r' with a common vertex v, either $r_{sv} = r'_{sv}$ or $r_{vt} = r'_{vt}$,

(iii) none of the networks of Figures 2 and 3 is embedded in G,

(iv) there is no bad configuration (no three routes r_1, r_2, r such that r_1 contains an edge e_1 not belonging to r_2, r_2 contains an edge e_2 not belonging to r_1 and r contains e_1 and e_2),

(v) G has only one edge or G is a result of

- *connecting two networks with linearly independent routes in parallel,*
- *connecting in series a network with linearly independent routes with a single edge network.*

Lemma 4. *A series-parallel network has linearly independent routes iff for every pair of different flows f_p^* for all sections p of r.*

Proof. In one direction: the networks of Fig 3 do not satisfy the property given in the lemma.

In the inverse direction by induction on the number of vertices using the composition operation for building networks with linearly independent routes. $\square$