S. Boldo et S. Le Roux Le 13 octobre 2004

Corrigé du TD de logique n°3

Déduction naturelle

Exercice 1 : ***Logique minimale***

Cf feuille précédente.

Exercice 2 : ***La déduction naturelle, c'est plus facile que Hilbert***

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \alpha\to\beta}\,\text{(hyp)} \quad \dfrac{\dfrac{}{\Gamma \vdash \gamma\to\alpha}\,\text{(hyp)} \quad \dfrac{}{\Gamma \vdash \gamma}\,\text{(hyp)}}{\Gamma \vdash \alpha}\,(\to\text{-E})}{\Gamma = (\alpha\to\beta), (\gamma\to\alpha), \gamma \vdash \beta}\,(\to\text{-E})}{(\alpha\to\beta), (\gamma\to\alpha) \vdash \gamma\to\beta}\,(\to\text{-I})}{(\alpha\to\beta) \vdash (\gamma\to\alpha)\to\gamma\to\beta}\,(\to\text{-I})}{\vdash (\alpha\to\beta)\to(\gamma\to\alpha)\to\gamma\to\beta}\,(\to\text{-I})$$

$$\dfrac{\dfrac{\dfrac{}{\alpha, \beta \vdash \alpha}\,\text{(hyp)}}{\alpha \vdash \beta\to\alpha}\,(\to\text{-I})}{\vdash \alpha\to\beta\to\alpha}\,(\to\text{-I})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \alpha\to\beta\to\gamma}\,\text{(hyp)} \quad \dfrac{}{\Gamma \vdash \alpha}\,\text{(hyp)}}{\Gamma \vdash \beta\to\gamma}\,(\to\text{-E}) \quad \dfrac{\dfrac{}{\Gamma \vdash \alpha\to\beta}\,\text{(hyp)} \quad \dfrac{}{\Gamma \vdash \alpha}\,\text{(hyp)}}{\Gamma \vdash \beta}\,(\to\text{-E})}{\Gamma = (\alpha\to\beta\to\gamma), (\alpha\to\beta), \alpha \vdash \gamma}\,(\to\text{-E})}{(\alpha\to\beta\to\gamma), (\alpha\to\beta) \vdash \alpha\to\gamma}\,(\to\text{-I})}{\alpha\to\beta\to\gamma \vdash (\alpha\to\beta)\to\alpha\to\gamma}\,(\to\text{-I})}{\vdash (\alpha\to\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma}\,(\to\text{-I})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \alpha\to\beta\to\gamma\to\delta}\,\text{(hyp)} \quad \dfrac{\dfrac{}{\Gamma \vdash \beta\to\alpha}\,\text{(hyp)} \quad \dfrac{}{\Gamma \vdash \beta}\,\text{(hyp)}}{\Gamma \vdash \alpha}\,(\to\text{-E})}{\Gamma \vdash \beta\to\gamma\to\delta}\,(\to\text{-E}) \quad \dfrac{}{\Gamma \vdash \beta}\,\text{(hyp)}}{\Gamma \vdash \gamma\to\delta}\,(\to\text{-E}) \quad \dfrac{\dfrac{}{\vdash \beta\to\gamma}\,\text{(hyp)} \quad \dfrac{}{\Gamma \vdash \beta}\,\text{(hyp)}}{\Gamma \vdash \gamma}\,(\to\text{-E})}{\Gamma = \alpha\to\beta\to\gamma\to\delta, \beta\to\alpha, \beta\to\gamma, \beta \vdash \delta}\,(\to\text{-E})}{\vdash (\alpha\to\beta\to\gamma\to\delta)\to(\beta\to\alpha)\to(\beta\to\gamma)\to\beta\to\delta}\,(\to\text{-I})^\star$$

$$\dfrac{\dfrac{\dfrac{}{\alpha \vee \beta \vdash \alpha \vee \beta}\,\text{(hyp)} \quad \dfrac{\dfrac{}{\alpha \vee \beta, \alpha \vdash \alpha}\,\text{(hyp)}}{\alpha \vee \beta, \alpha \vdash \beta \vee \alpha}\,(\vee\text{-I-2}) \quad \dfrac{\dfrac{}{\alpha \vee \beta, \beta \vdash \beta}\,\text{(hyp)}}{\alpha \vee \beta, \beta \vdash \beta \vee \alpha}\,(\vee\text{-I-1})}{\alpha \vee \beta \vdash \beta \vee \alpha}\,(\vee\text{-E})}{\vdash \alpha \vee \beta\to\beta \vee \alpha}\,(\to\text{-I})$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \alpha \vee \gamma}\,\text{(hyp)} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, \alpha \vdash \alpha\to\beta}\,\text{(hyp)} \quad \dfrac{}{\Gamma, \alpha \vdash \alpha}\,\text{(hyp)}}{\Gamma, \alpha \vdash \beta}\,(\to\text{-E})}{\Gamma, \alpha \vdash \beta \vee \gamma}\,(\vee\text{-I-1}) \quad \dfrac{\dfrac{}{\Gamma, \gamma \vdash \gamma}\,\text{(hyp)}}{\Gamma, \gamma \vdash \beta \vee \gamma}\,(\vee\text{-I-2})}{\Gamma = (\alpha\to\beta), (\alpha \vee \gamma) \vdash \beta \vee \gamma}\,(\vee\text{-E})}{\vdash (\alpha\to\beta)\to(\alpha \vee \gamma)\to(\beta \vee \gamma)}\,(\to\text{-I})^\star$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \alpha \vee (\beta\wedge\gamma)}\,(\text{hyp}) \quad \dfrac{\dfrac{}{\Gamma, \alpha \vdash \alpha}\,(\text{hyp})}{\Gamma, \alpha \vdash \alpha \vee \beta}\,(\vee\text{-I-1}) \quad \dfrac{\dfrac{}{\Gamma, \beta\wedge\gamma \vdash \beta\wedge\gamma}\,(\text{hyp})}{\Gamma, \beta\wedge\gamma \vdash \beta}\,(\wedge\text{-E-1})}{\Gamma, \beta\wedge\gamma \vdash \alpha \vee \beta}\,(\vee\text{-I-2})}{\Gamma \vdash \alpha \vee \beta}\,(\vee\text{-E}) \quad \dfrac{idem}{\Gamma \vdash (\alpha \vee \gamma)}\,()}{\Gamma = \alpha \vee (\beta\wedge\gamma) \vdash (\alpha \vee \beta)\wedge(\alpha \vee \gamma)}\,(\wedge\text{-I})}{\vdash \alpha \vee (\beta\wedge\gamma)\rightarrow(\alpha \vee \beta)\wedge(\alpha \vee \gamma)}\,(\rightarrow\text{-I})$$

Exercice 3 : ***Affaiblissement***

C'est évident : il suffit de rajouter à tous les jugements de l'arbre de démonstration de $\Gamma \vdash \varphi$ les hypothèses de Δ et on obtient un arbre de démonstration de $\Gamma, \Delta \vdash \varphi$.

Exercice 4 : ***Quelques démonstrations en déduction naturelle***

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p\wedge(q\wedge r) \vdash p\wedge(q\wedge r)}\,(\text{hyp})}{p\wedge(q\wedge r) \vdash p}\,(\wedge\text{-E-1}) \quad \dfrac{\dfrac{\dfrac{}{p\wedge(q\wedge r) \vdash p\wedge(q\wedge r)}\,(\text{hyp})}{p\wedge(q\wedge r) \vdash q\wedge r}\,(\wedge\text{-E-2})}{p\wedge(q\wedge r) \vdash q}\,(\wedge\text{-E-1})}{p\wedge(q\wedge r) \vdash (p\wedge q)}\,(\wedge\text{-I}) \quad \dfrac{\dfrac{\dfrac{}{p\wedge(q\wedge r) \vdash p\wedge(q\wedge r)}\,(\text{hyp})}{p\wedge(q\wedge r) \vdash q\wedge r}\,(\wedge\text{-E-2})}{p\wedge(q\wedge r) \vdash r}\,(\wedge\text{-E-2})}{p\wedge(q\wedge r) \vdash (p\wedge q)\wedge r}\,(\wedge\text{-I})}{\vdash p\wedge(q\wedge r)\rightarrow(p\wedge q)\wedge r}\,(\rightarrow\text{-I})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p\vee(q\wedge r) \vdash p\vee(q\wedge r)}\,(\text{hyp}) \quad \dfrac{\dfrac{}{p\vee(q\wedge r), p \vdash p}\,(\text{hyp})}{p\vee(q\wedge r), p \vdash p\vee q}\,(\vee\text{-I-1}) \quad \dfrac{\dfrac{\dfrac{}{p\vee(q\wedge r), q\wedge r \vdash q\wedge r}\,(\text{hyp})}{p\vee(q\wedge r), q\wedge r \vdash q}\,(\wedge\text{-E-1})}{p\vee(q\wedge r), q\wedge r \vdash p\vee q}\,(\vee\text{-I-2})}{p\vee(q\wedge r) \vdash p\vee q}\,(\vee\text{-E}) \quad \dfrac{idem}{p\vee(q\wedge r) \vdash p\vee r}}{p\vee(q\wedge r) \vdash (p\vee q)\wedge(p\vee r)}\,(\wedge\text{-I})}{\vdash p\vee(q\wedge r)\rightarrow(p\vee q)\wedge(p\vee r)}\,(\rightarrow\text{-I})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p\vee\neg p}\,(\text{hyp}) \quad \dfrac{}{\Gamma, p \vdash p}\,(\text{hyp}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg p \vdash (p\rightarrow q)\rightarrow p}\,(\text{hyp}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg p, p \vdash \neg p}\,(\text{hyp}) \quad \dfrac{}{\Gamma, \neg p, p \vdash p}\,(\text{hyp})}{\Gamma, \neg p, p \vdash \bot}\,(\rightarrow\text{-E})}{\Gamma, \neg p, p \vdash q}\,(\textit{e.f.q.s.})}{\Gamma, \neg p \vdash p\rightarrow q}\,(\rightarrow\text{-I})}{\Gamma, \neg p \vdash p}\,(\rightarrow\text{-E})}{\Gamma = p\vee\neg p, (p\rightarrow q)\rightarrow p \vdash p}\,(\vee\text{-E})}{p\vee\neg p \vdash ((p\rightarrow q)\rightarrow p)\rightarrow p}\,(\rightarrow\text{-I})}{\vdash (p\vee\neg p)\rightarrow(((p\rightarrow q)\rightarrow p)\rightarrow p)}\,(\rightarrow\text{-I})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg q}\,(\text{hyp}) \quad \dfrac{\dfrac{}{\Gamma \vdash p\rightarrow q}\,(\text{hyp}) \quad \dfrac{}{\Gamma \vdash p}\,(\text{hyp})}{\Gamma \vdash q}\,(\rightarrow\text{-E})}{\Gamma = p\rightarrow q, \neg q, p \vdash \bot}\,(\rightarrow\text{-E})}{p\rightarrow q, \neg q \vdash \neg p}\,(\rightarrow\text{-I})}{p\rightarrow q \vdash \neg q\rightarrow\neg p}\,(\rightarrow\text{-I})}{\vdash (p\rightarrow q)\rightarrow(\neg q\rightarrow\neg p)}\,(\rightarrow\text{-I})$$

$$\dfrac{\dfrac{\dfrac{\textit{prémisse}}{\vdash \Gamma \vdash \neg q} \quad \dfrac{\textit{prémisse}}{\vdash \Gamma \vdash q}}{\Gamma \vdash \bot}\,(\rightarrow\text{-E})}{\Gamma \vdash p}\,(\textit{e.f.q.s.})$$

Exercice 5 : ***Déduction naturelle classique***

$$\dfrac{\dfrac{\dfrac{\neg\neg p, \neg p \vdash \neg p \to \bot \qquad \neg\neg p, \neg p \vdash \neg p}{\neg\neg p, \neg p \vdash \bot}\,{\to}E}{\neg\neg p \vdash p}\,RAA}{\vdash \neg\neg p \to p}\,{\to}I$$

$$\dfrac{\Gamma, \neg p \vdash \bot}{\Gamma \vdash p}\,RAA \qquad\qquad \dfrac{\dfrac{\Gamma, \neg p \vdash p \to \bot \qquad \Gamma, \neg p \vdash p}{\Gamma, \neg p \vdash \bot}\,{\to}E}{\Gamma \vdash p}\,RAA$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(p \vee \neg p), p \vdash p \vee \neg p \to \bot \qquad \dfrac{\neg(p \vee \neg p), p \vdash p}{\neg(p \vee \neg p), p \vdash p \vee \neg p}\,{\vee}I_g}{\neg(p \vee \neg p), p \vdash \bot}\,{\to}E}{\neg(p \vee \neg p) \vdash \neg p}\,{\to}I}{\neg(p \vee \neg p) \vdash p \vee \neg p}\,{\vee}I_g}{\vdash p \vee \neg p}\,precedent$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, p, \neg q \vdash \neg q \to \neg p \qquad \Gamma, p, \neg q \vdash \neg q}{\Gamma, p, \neg q \vdash \neg p}\,{\to}E \quad \Gamma, p, \neg q \vdash p}{\Gamma, p, \neg q \vdash \bot}\,{\to}E}{\Gamma, p \vdash q}\,RAA}{\Gamma \vdash p \to q}\,{\to}I$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg p, (p \to q) \to p \vdash \neg p \qquad \dfrac{\neg p, (p \to q) \to p \vdash (p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg p, (p \to q) \to p, p \vdash \neg p \qquad \neg p, (p \to q) \to p, p \vdash p}{\neg p, (p \to q) \to p, p \vdash \bot}\,{\to}E}{\neg p, (p \to q) \to p, p \vdash q}\,{\bot}E}{\neg p, (p \to q) \to p \vdash p \to q}\,{\to}I}{\neg p, (p \to q) \to p \vdash p}\,{\to}E}{\neg p, (p \to q) \to p \vdash \bot}\,{\to}E}{\vdash \neg p \to \neg((p \to q) \to p)}\,{\to}I*}{\vdash ((p \to q) \to p) \to p}\,contrap$$

$\Gamma_1 = \Gamma, \neg(p \wedge q)$ et $\Gamma_2 = \Gamma_1, p, q$ et $\Gamma_3 = \Gamma_1, p, \neg q$ et $\Gamma_4 = \Gamma_1, \neg p$

$$\dfrac{\dots}{\dots}\quad \dfrac{\Gamma_2 \vdash p \qquad \Gamma_2 \vdash q}{\Gamma_2 \vdash p \wedge q}\,{\wedge}I}{\Gamma_2 \vdash \bot}\,{\to}E}{\Gamma_2 \vdash r}\,{\bot}E \quad \dfrac{\Gamma_3 \vdash \neg q}{\Gamma_3 \vdash \neg p \vee \neg q}\,{\vee}I_d \quad \dfrac{}{\dots}\,H}{\Gamma_3 \vdash r}\,{\to}E \quad \dfrac{}{? \vdash q \vee \neg q}\,te}{\Gamma_1, p \vdash r}\,{\vee}E \quad \dfrac{\dfrac{\Gamma_4 \vdash \neg p}{\Gamma_4 \vdash \neg p \vee \neg q}\,{\vee}I_g \quad \dfrac{}{\dots}\,H}{\Gamma_4 \vdash r}\,{\to}E \quad \dfrac{}{\Gamma_1 \vdash p \vee \neg p}\,te}{\Gamma_1 \vdash r}\,{\vee}E$$

Soit $\Gamma_1 = \Gamma, \neg(p \vee q)$ et $\Gamma_2 = \Gamma, p$ et $\Gamma_3 = \Gamma_1, \neg p, q$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots \quad \dfrac{\Gamma_2 \vdash p}{\Gamma_2 \vdash p \vee q}\,\vee I_g}{\Gamma_2 \vdash \bot}\,\to E}{\Gamma_2 \vdash r}\,\bot E \quad \dfrac{\dfrac{\dfrac{\dots \quad \dfrac{\Gamma_3 \vdash q}{\Gamma_3 \vdash p \vee q}\,\vee I_d}{\Gamma_3 \vdash \bot}\,\to E}{\Gamma_3 \vdash r}\,\bot E \quad \dfrac{}{\Gamma_1, \neg p, \neg q \vdash r}\,H \quad \dfrac{}{\Gamma_1, \neg p \vdash q \vee \neg q}\,te}{\Gamma_1, \neg p \vdash r}\,\vee E \quad \dfrac{}{\Gamma_1 \vdash p \vee \neg p}\,te}{\Gamma_1 \vdash r}\,\vee E$$

On pose $\Gamma_1 = \Gamma, \neg(p \to q)$, $\Gamma_2 = \Gamma_1, p$, $\Gamma_3 = \Gamma_2, q$, $\Gamma_4 = \Gamma_1, \neg p$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots \quad \dfrac{}{\Gamma_3 \vdash p \to q}\,\to I}{\Gamma_3 \vdash \bot}\,\to E}{\Gamma_3 \vdash r}\,\bot E \quad \dfrac{}{\Gamma_2, \neg q \vdash r}\,H \quad \dfrac{}{\Gamma_2 \vdash q \vee \neg q}\,te}{\Gamma_2 \vdash r}\,\vee E \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots \quad \dfrac{\Gamma_4, p \vdash \neg p \quad \Gamma_4, p \vdash p}{\Gamma_4, p \vdash \bot}\,\to E}{\Gamma_4, p \vdash q}\,\bot E}{\Gamma_4 \vdash p \to q}\,\to I}{\Gamma_4 \vdash \bot}\,\to E}{\Gamma_4 \vdash r}\,\bot E \quad \dfrac{}{\Gamma_1 \vdash p \vee \neg p}\,te}{\Gamma_1 \vdash r}\,\vee E$$