Chapter 3

Proof Nets

As is the case for classical or intuitionistic sequent calculus, a proof of a sequent $\Gamma \vdash \Delta$ (or $\vdash \Gamma^{\perp}, \Delta$ in our asymmetric presentation) can contain a lot of details that sometimes are uninteresting. For example, consider how many uninterestingly different ways there are to form a proof of $\vdash \Gamma, (A_1 \parr A_2), \ldots, (A_{n-1} \parr A_n)$ starting from a derivation of $\vdash \Gamma, A_1, A_2, \ldots, A_n$.
This unpleasant fact derives from the sequential nature of proofs in sequent calculus: if we want to apply a set S of n rules to different parts of a sequent, we cannot apply them in one step, even if they do not interfere with each other! We must *sequentialize* them, i.e. choose a linear order on S and apply the rules in n steps, according to this order. This phenomenon can lead us to $n!$ proofs that are intensionally different, but that we would like to consider as just one proof.
At this point, there is a natural question that needs to be answered: "is there a representation of proofs that abstracts from such uninteresting details?". A similar question is answered positively in the case of Intuitionistic Sequent Calculus by means of what is known, after Prawitz, as *Natural Deduction*.

3.1 Digression: Natural Deduction for ISC

Let's recall briefly the natural deduction presentation of the implication-only fragment of Intuitionistic Sequent Calculus, and some of its fundamental properties. For details, the interested reader ought to refer to [GLT90].

Definition 3.1.1 (Natural deduction)

- *A is a deduction of the formula A from hypothesis A*
- *if $\dfrac{\mathsf{D}}{B}$ is a deduction of B from a multiset of hypothesis Γ, then*

$$\dfrac{\dfrac{\mathsf{D}'}{B}}{A \Rightarrow B}$$

is a deduction of $A \Rightarrow B$ from hypothesis Γ', where Γ' is Γ, eventually without some occurrences of A.

- *if* $\dfrac{\mathsf{D}_1}{A \Rightarrow B}$ *is a deduction of* $A \Rightarrow B$ *from hypothesis* Γ *and* $\dfrac{\mathsf{D}_2}{A}$ *is a deduction of* A *from hypothesis* Γ'*, then*

$$\dfrac{\dfrac{\mathsf{D}_1}{A \Rightarrow B} \quad \dfrac{\mathsf{D}_2}{A}}{B}$$

is a deduction of B *from hypothesis* $\Gamma \uplus \Gamma'$*, where* $\uplus$ *is multiset union.*

It is possible to give a simple translation from any proof P in LJ to a natural deduction D, and to show that this translation is surjective (see[GLT90]), so that the two systems are equivalent as far as provability is concerned (i.e. proving a sequent $\Gamma \vdash A$ (recall that we are talking about Intuitionistic calculus) is equivalent to build a natural deduction of A with premisses taken (as much times as we want) from Γ). See [GLT90] for more details.
The natural deduction tree structure D associated to a proof P can be thinked of as the quotient of all the proofs that can be obtained from P by legal permutations of the rules in P.
The cut elimination property for ISC becomes a similar property still called cut elimination (but where a cut is defined in a different way) for Natural Deductions. A natural deduction tree can be put in exact correspondence with typed λ-calculus (that can be considered as just a notation for such deductions), and the cut-elimination in Natural Deduction corresponds exactly to β-reduction in λ-calculus.
In fact, we can present the simple typed λ-calculus in a way that mimicks directly the construction of a derivation tree.

Definition 3.1.2 (Simple typed λ-calculus)

- *x:A is a variable of type A*
- *if* $\dfrac{\mathsf{D}}{M : B}$ *is a deduction of M:B from a multiset of hypotheses* Γ*, then*

$$\dfrac{\dfrac{\mathsf{D}'}{M : B}}{(\lambda x : A.M) : A \Rightarrow B}$$

is a deduction of $(\lambda x : A.M) : A \Rightarrow B$ *from hypothesis* Γ'*, where* Γ' *is* Γ*, where all occurrences of x:A are discarded.*

- *if* $\dfrac{\mathsf{D}_1}{M : A \Rightarrow B}$ *is a deduction of M:A⇒B from hypothesis* Γ *and* $\dfrac{\mathsf{D}_2}{N : A}$ *is a deduction of N:A from hypothesis* Γ'*, then*

$$\dfrac{\dfrac{\mathsf{D}_1}{M : A \Rightarrow B} \quad \dfrac{\mathsf{D}_2}{N : A}}{(MN) : B}$$

is a deduction of (MN):B from hypothesis $\Gamma \uplus \Gamma'$*, where* $\uplus$ *is multiset union, if the declaration of types of common variables agree.*

From a simple typed λ-term it is possible to recover the full structure of the derivation of its well-typing. This is why such terms are also used as a "notation" for derivations in natural deduction.

Remark 3.1.3 *The rules for building a natural deduction tree have the following properties:*

- **global well-formedness:** the application of a building rule can require the inspection of the global tree already built (see the case for $\Rightarrow$).
- **local correctness:** if all the rules are applied properly (i.e. the proof tree is well-formed), then the deduction is correct.

3.1.1 Natural Deduction and Proof Structures

As already noted, a natural deduction is essentially a tree, with just one root that correspond to the just one conclusion in an intuitionistic sequent.
Can we adapt this construction to the case of Linear Logic? We have a first difficulty to face: sequents in LL, even in the symmetric presentation, have more than one conclusion, so that it seems not clear how to associate a *tree* to a class of derivations. Actually, it is possible to adapt natural deduction to this situation by using an *interpretation* of symmetric linear sequents in terms of intuitionistic linear sequents of ILL. Anyway, there is a more natural way (and much more interesting, as we will see later) to build a mathematical object that identifies uninterestingly different derivations in Linear Sequent Calculus.
We will consider no longer trees, but *graphs*, as suggested by the asymmetric presentation of *LSC*. For simplicity, the presentation that follows is restricted to the case of the multiplicative fragment *MLL* of linear logic.

3.1.2 Proof Structures. Concrete and abstract form.

Let's start by defining the notion of *sequential proof structure*, or *SPS*, by following exactly the construction of derivations in the asymmetric sequent calculus for *MLL*.

Definition 3.1.4 (sequential proof structure) *Define inductively "*S is a *SPS* with conclusions $X_1, \ldots, X_n$*", where $X_1, \ldots, X_n$ are multiplicative formulae, as follows:*

1. **ax link.** *For any multiplicative formula A*

$$\frac{}{A \quad A^{\perp}}$$

   is a SPS *with conclusions $A, A^{\perp}$.*

2. **par link.** *If S_1 is a* SPS *with conclusions $X_1, \ldots, X_n, A, B$, then S is a* SPS *with conclusions $X_1, \ldots, X_n, A \parr B$, where S is :*

$$\frac{\begin{array}{c} S_1 \\ \vdots \\ A \quad B \end{array}}{A \parr B}$$

3. **tensor link .** *If S_1 is a* SPS *with conclusions $X_1, \ldots, X_n, A$, and S_2 is a* SPS *with conclusions $Y_1, \ldots, Y_m, B$,*

$$\dfrac{\begin{array}{cc} S_1 & S_2 \\ \vdots & \vdots \\ A & B \end{array}}{A \otimes B}$$

is a SPS *with conclusions $X_1, \ldots, X_n, Y_1, \ldots, Y_m, A \otimes B$*

4. **cut link .** *If S_1 is a* SPS *with conclusions $X_1, \ldots, X_n, A$, and S_2 is a* SPS *with conclusions $Y_1, \ldots, Y_m, A^\perp$,*

$$\underline{\begin{array}{cc} S_1 & S_2 \\ \vdots & \vdots \\ A & A^\perp \end{array}}$$

is a SPS *with conclusions $X_1, \ldots, X_n, Y_1, \ldots, Y_m$*

This definition allows to associate in a straightforward way an *SPS* to any derivation of an *MLL* sequent.

Exercise 3.1.5 (Embedding sequents into *SPS*'s) *Give a formal translation of a derivation of a sequent $\vdash A_1, \ldots, A_n$ into a* SPS *with conclusions $A_1, \ldots, A_n$.*

What is more interesting, we can prove that *any SPS* actually comes from a derivation of a sequent in *SPS*.

Theorem 3.1.6 (Correction of *SPS*'s) *If S is an* SPS *with conclusions $A_1, \ldots, A_n$, then there exists a derivation D in MLL of the sequent $\vdash A_1, \ldots, A_n$. Furthermore, the translation of this derivation is S.*

Proof. Exercise. □

Are these *SPS*'s the structures we were looking for in order to abstract from the too many details present in the derivations of the sequent calculus? Apparently yes: in a *SPS* there is no longer a trace of the order of application of non-interfering rules. Unfortunately, *SPS*'s enjoy the same properties of natural deduction, i.e. the *local correctness* and *global well-formedness*, and here to check the well-formedness of a **tensor** link in a *SPS S we have to find a possible order of construction of S*.
So, *SPS*'s are easy to build, but it seems to be very difficult to check that a structure looking like an *SPS* is actually an *SPS*, and not simply what is called a *concrete proof structure* (or *CPS*), that is a composition of links choosen among

$$\overline{A \quad A^\perp} \qquad \dfrac{A \quad B}{A \parr B} \qquad \dfrac{A \quad B}{A \otimes B} \qquad \underline{A \quad A^\perp}$$

and satisfying the conditions:

- every formula is conclusion of exactly one link
- every formula is premiss of at most one link.

Remark 3.1.7 (proof-structures) *In the literature,* CPS *are ususally called* pré-réseaux *or* proof structures*, while* SPS *are called* réseaux *or* proof-nets.

In the concrete form, the horizontal segments represent what are called *links*. Formulae directly above a link are called *premisses* of the link, while those directly below are called *conclusions* of the link. The formulae that are not premisses of any link in a *CPS* are called conclusions of that *CPS*. We also call *cut-formulae* those formulae that are premisses of a **cut** link, and *terminal formulae* of a *CPS* the cut-formulae and the conclusions of that *CPS*.

3.2 Correctness Criteria

We need to look for a criterion of correctness to ensure that a given *CPS* is actually an *SPS*, i.e. represents a derivation in *MLL*. We will provide here a rather detailed account of the most appealing criterions known to date (essentially due to Vincent Danos and Laurent Regnier, see [Reg91, Dan90]), and a quick survey of the other ones we know of. Essentially this will lead us to a tour of the following criteria:

Topological criteria

Long-trip [Gir87]

Contractibility [Dan90]

Acyclic-Connected [Dan90]

Hereditary Secessiveness [Dan90]

Graph Parsing [GMM97]

Deadlock-freeness [AD]

For each of them a proof of *soundness* and *completeness* can be found in the referenced papers. Here we mean by soundness the implication *if a* CPS *is an* SPS *than it satisfies the criterion*, and by completeness the reverse implication *if a* CPS *satisfies the criterion, then it is an* SPS. Typically, soundness can be shown in a straightforward way by induction on the structure of an *SPS*, while the completeness requires much more work. We will detail in what follows the first two criteria and show their soundness and completeness in a way that factors the difficult implication, and will give us two criteria for the price of one. What follows is an original elaboration of Chapter 6, 7, and 8 of [Dan90].

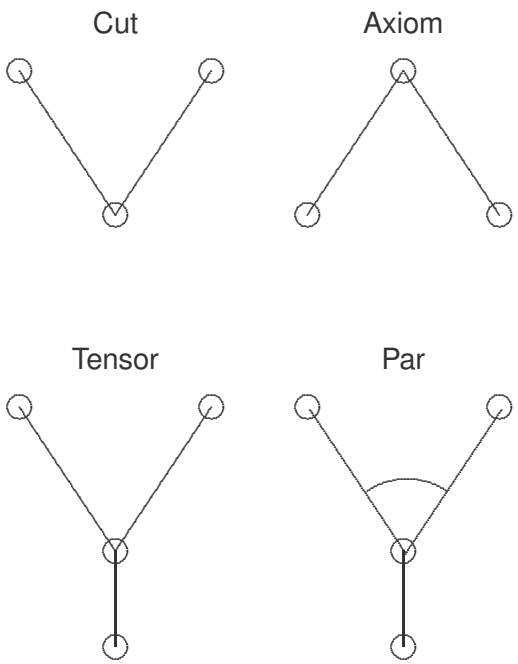
3.2.1 Topological Criteria

Let's start our survey by identifying the underlying topological structure of a *CPS*. To do so, we abstract from the concrete details of the links of a *CPS*, and we immediately notice that every link is essentially a connective with two (**ax** and **cut** link) or three (**par** and **tensor** links) connecting

ports. The **ax** and **cut** connectives directly connect their two ports, while the **par** and **tensor** connectives connect two of the ports (the premisses) with a main port (the conclusion).
This observation suggests to represent abstractly each of these links by two connected segments in a graph.

Definition 3.2.1 *A* paired graph *is any undirected graph $S = (E, V, C)$ (where E is a set of* edges*, and V is a set of* vertices*), equipped with a set $\mathsf{C}(S)$ of pairwise disjoint couples of co-incident edges*[1]*.The set of pairs $\mathsf{C}(S)$ can also be seen as a binary relation on $E \times E$.*

Let's call simply *pairs* such couples of edges and *basis of a pair* a node common to the two edges, and let's say that two edges that belong to a same couple are *coupled*.
Graphically we denote the fact that two edges are coupled by a little arc tying them.
It is then straightforward to associate a paired graph S^- to a *CPS* S: just write an axiom link and a cut link as two edges, a **tensor** link as three edges and a **par** link as three edges, but where the two upper ones are *coupled* as in the following picture.



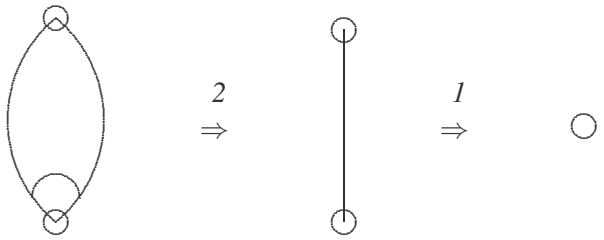
Now we can state our first correctness criterion:

Definition 3.2.2 (Acyclic-Connected) *Given a paired graph G, let $GC(G)$ be the set of graphs that can be obtained from it by removing exactly one edge for each pair of coupled edges.*
*We say that that G satisfies the Acyclic-Connected condition (*ACC*) if all the graphs in $GC(G)$ are acyclic and connected (i.e. they are trees). We will say that a* CPS *S satisfies* ACC *if S^- satisfies* ACC.

[1] Two edges are co-incident if they have at least one node in common.

As we will see later, a *CPS* satisfying *ACC* is a *SPS*, but this criterion invites to an exponential check: the graphs in $GC(S)$ are 2^p where p is the number of **par** link in the *CPS* S.
It is possible to do much better than that.
Let's consider the following notion of graph reduction.

Definition 3.2.3 *We define the following graph rewriting rules on paired graphs:*



with the condition that:

- *rule 1 can be applied only to non-coupled edges connecting two different nodes*
- *rule 2 applies only to two coupled edges that connect the same two* distinct *nodes.*

Exercise 3.2.4 *Show that the graph reduction just defined is strongly normalizing and weakly confluent (hence confluent).*

Definition 3.2.5 (Contractibility) *A* CPS S *is* contractible *iff* S^- *reduces to a single node.*

Again, we will see later that a contractible *CPS* is an *SPS*. A check for contractibility, though, can be performed in at most quadratic time.
Now it is very easy to show the sequence of implications:

$$S \text{ is a } \textit{SPS} \Rightarrow S \text{ is contractible} \Rightarrow S \text{ satisfies } \textit{ACC}.$$

The proofs are left as exercises.

Exercise 3.2.6 (*SPS*'s are contractibles) *Show by induction on definition 3.1.4 that a* SPS *is contractible.*

Exercise 3.2.7 (Contractible *CPS*'s satisfy *ACC*) *Let S be a paired graph reducing to S' by rule 1 or 2 above.*

- *Show that S satisfies* ACC *iff S' does.*
- *Infer from this that if S is contractible then S satisfies* ACC.

3.2.2 Completeness of the criteria

The two implications just proved tell us that both criteria are *sound*. Furthermore, we know by the second implication that *completeness* of the *ACC* criterion implies completeness of the contractibility criterion.
So we are left to show that a *CPS* satisfying *ACC* is indeed an *SPS*. To do so, we will again procede by induction on the structure of the graph, but a key point will be the possibility of splitting a given paired graph satisfying *ACC* into two smaller paired graphs that still satisfy *ACC*. Unlike what happens in the case of the proof of completeness of the original *long-trip* criterion, we will split the graph in correspondence of a *par* link, and not of a *tensor* link.

Definition 3.2.8 (splitting pair) *Let S be a paired graph. We say S is* split *if there is a pair p of S s. t. the basis of p is disconnected from the other two extremities of p in $S \setminus p$. Such pair is called* splitting.

Exercise 3.2.9 (splitting preserves *ACC*) *Let S be a paired graph split by p; show that if S satisfies* ACC *then $S \setminus p$ has two connected components, also satisfying* ACC. *Show the converse.*

Is it always possible to find a splitting pair in a paired graph that satisfies ACC? Yes, it is! But here we find the only hard part of the proof.

Theorem 3.2.10 (Paired graphs with at least one pair are split) *Let S be a* finite *paired graph satisfying* ACC. *If S counts at least one pair then it is* split.

Proof. This is a difficult theorem: see [Dan90], Proposition 4-5, for details. See also the Appendix. □

Exercise 3.2.11 *Show a counter example to the theorem above in case one allows infinite graphs.*

Now we have all the necessary tools to prove the completeness of the *ACC* criterion, and hence of the Contractibility criterion too.

Theorem 3.2.12 (Completeness) *Any* CPS S *satisfying* ACC *is an* SPS.

Proof. We will procede by induction on the size of S, but the inductive steps will force us to look at S not just as a *CPS* or a *SPS*, but as a slightly more general structure.

Exercise 3.2.13 *Add to definition 3.1.4 the following clause:*

1'. **hypothesis-link.** *For any multiplicative formula A*

$$\overline{A}$$

is a SPS *with hypotheses with conclusions A.*

to have the definition of a SPS *with hypotheses. Similarly extend the definition of* CPS *to cope with hypotheses (this is only induction loading).*
Now, let S be a CPS *w. h. satisfying* ACC*:*

- *if S has no pairs show by induction on the number of $\otimes$ that S is a* SPS*w.h.,*
- *if S has a pair use the non obvious graph-theoretic property above to show by induction on the number of $\parr$ that S is a* SPS*w.h..*

□

Exercise 3.2.14 (Danos-Schellinx) *Let A be a provable multiplicative formula.*

1. *Suppose S is a cut-free proof-net proving A, so it consists in the "tree" of A, say $T(A)$, together with a set of axiom links over its leaves (the atoms of A). Show that $C(A) = N(A)+1$, where $C(A)$ is the number of connected components of $T(A)$ under any switching s, and $N(A)$ is the number of axiom links in S.*
2. *Prove that $P(A) = N(A)$*
3. *infer from this that $\vdash A, A, \cdots, A$; and better that if A_1, A_2, ..., A_n are provable (for $n > 1$), then $\vdash A_1, A_2, \cdots, A_n$*
4. *again infer from point 1 that if $P(A) + N(A) \equiv P(B) + N(B)$ (mod 2), then $\vdash A, B$.*

3.2.3 Handling the MIX rule

In the proof of correctness for the ACC criterion, the only difficult part really uses (as you can see in the Appendix), a weaker property, called Weak-ACC or just AC, that asks the correction graphs to be acyclic, but not necessarily connected. This allows to capture CPS that are *families* of SPS's, and correspond to proof in *MLL* extended with the following MIX rule:

$$\frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{R}MIX$$

The MIX rule allows for example to prove the linear implication

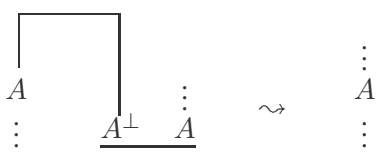
$$A \otimes B \multimap A \parr B$$

It is then a simple exercise to adapt the definitions and proofs just seen to obtain the following

Theorem 3.2.15 (AC criterion) *A CPS satisfying AC is a family of SPS's, and represent a proof in MLL+ MIX.*

3.3 Reduction of proof-nets.

One can reduce a proof-net either this way (axiom move):



or that way (multiplicative move):

Exercise 3.3.1 (Reduction preserves correctness)

1. *Show that the axiom-move preserves correctness.*
2. *Call R the lhs of the multiplicative-move; take S to be any switch of R and call S' the graph obtained from S by erasing the three links $\otimes$, $\parr$ and cut concerned by the move. Show that among A, B, $A^\perp$ and $B^\perp$ the only two connected vertices are $A^\perp$ and $B^\perp$.*
3. *Deduce from the previous point that the multiplicative move also preserves correctness.*

So these two moves preserve correctness, and the right handside is indeed a proof-net if the left hand side is one.

Exercise 3.3.2 (Confluence and Strong Normalisation)

1. *Show that this rewriting process is noetherian;*
2. *show it is confluent (by the previous point it is enough to prove that it is locally confluent) and every reduction to normal form have the same length.*

Exercise 3.3.3 (splitting $\otimes$) *Let S be a* CPS*, a $\otimes$ in S is said to split it if its erasure splits S in two connected components.*

1. *show that in a* CPS *satisfying* ACC*, with no terminal $\parr$-links, there is such a $\otimes$ (unless the structure is an axiom link).*
2. **(Andreoli-Pareschi)** *show that under the same hypotheses one can even find an hereditarily splitting $\otimes$, i.e., a splitting $\otimes$, the premises of which are $\parr$'s or again hereditarily splitting $\otimes$.*

3.4 Proof Nets for MELL

We have seen up to now what are the proof nets for *MLL*, whose behaviour is fully satisfactory. For *MELL*, we can proceed to a similar construction, but the traditional presentation of exponentials, as we will see, is less satisfactory. Here follows the traditional presentation of Proof Nets for *MELL*:

Definition 3.4.1 *The set of proof nets, denoted by PN, is the smallest set satisfying the following properties:*