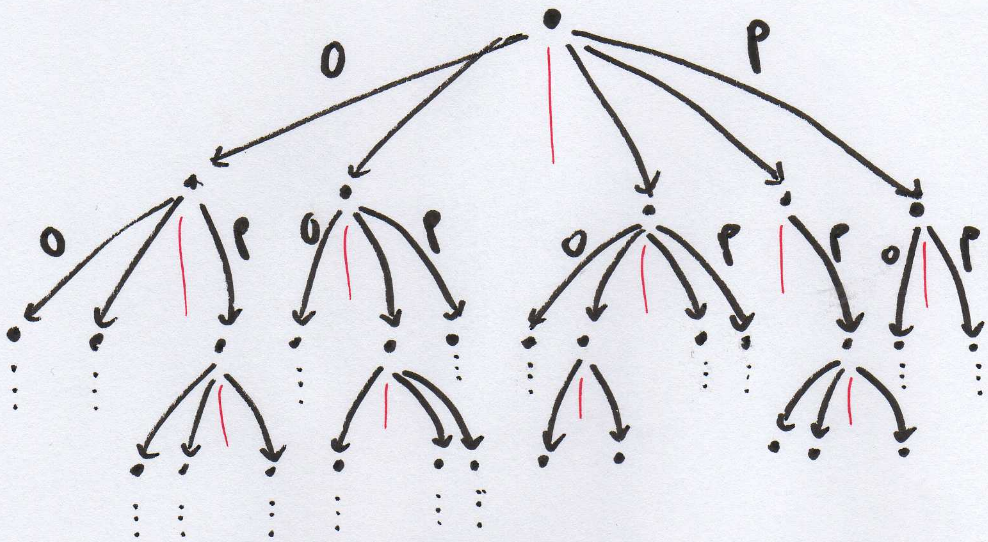
Conway Games

A Conway game is a tree:



* each node is a <u>position</u>
* each edge is a <u>move</u>
* a <u>play</u> is a path from the root

By convention, moves pointing left are by <u>Opponent</u>; and moves pointing right are by <u>Player.</u>

Formally, a Conway game is a triple

$$G \triangleq \langle M_G, \lambda_G, P_G \rangle$$

where

* M_G is a (countable) set of <u>moves</u>
* $\lambda_G : M_G \to \{O, P\}$ is the <u>labelling</u> function
* $P_G \subseteq M_G^*$ is a set of words/strings over M_G satisfying

  (p1) ε, the empty string, is in P_G

  (p2) P_G is prefix-closed

The set P_G specifies the game tree.

<u>Ex</u> $\mathbb{1} \triangleq \langle \emptyset, \emptyset, \{\varepsilon\} \rangle$, the empty game

$$\perp \triangleq \langle \{q\}, q \mapsto O, \{\varepsilon, q\} \rangle$$

$$\perp^* \triangleq \langle \{q\}, q \mapsto O, \{\varepsilon, q, qq, qqq, \cdots\} \rangle$$

$$\mathrm{bool} \triangleq \langle \{q, tt, ff\}, \begin{array}{c} q \mapsto O \\ tt, ff \mapsto P \end{array}, \{\varepsilon, q, q\,tt, q\,ff\} \rangle$$

Conway games are closed under:

- negation (exchange of O and P)
- tensor (interleaving)

Formally, we define

$$G^{\perp} \triangleq \langle M_G, \overline{\lambda}_G, P_G \rangle \quad \text{where}$$

$$\overline{\lambda}_G(m) = O \quad \text{iff} \quad \lambda_G(m) = P$$
$$\ldots \quad P \quad \text{iff} \quad \ldots \quad O$$

&

$$G \otimes H \triangleq \langle M_G + M_H, [\lambda_G, \lambda_H], ? \rangle \quad \text{where}$$

$$? = \{ s \in (M_G + M_H)^* \mid s\restriction_G \in P_G \wedge s\restriction_H \in P_H \}$$

We will see more such constructors later...

<u>Qu</u> What is $\mathbb{1}^{\perp} \otimes \mathbb{1}$?

$(\mathbb{1}^*)^{\perp} \otimes \mathbb{1}^*$?

$\vdots$

A <u>strategy</u> σ for a Conway game G prescribes the behavior of P as a function of (the play to date and) the behavior of O.

Formally, $\sigma \subseteq P_G$ satisfying

(s1) $\varepsilon \in \sigma$
(s2) all $s \in \sigma$ alternate
(s3) ... start with an O-move
(s4) ... end with a P-move
(s5) closed under P-ending prefixes
(s6) deterministic [$sa, sb \in \sigma \Rightarrow a = b$]

<u>Ex</u> Only one strategy, $\{\varepsilon\}$, for $\mathbf{1}$
Two strategies, $\{\varepsilon\}$ and $\{\varepsilon, qq\}$ for $\mathbb{1}^\perp \otimes \mathbb{1}$
Many strategies for $(\mathbb{1}^*)^\perp \otimes \mathbb{1}^*$ ...

<u>Qu</u> What about $\mathrm{bool}^\perp \otimes \mathrm{bool}$?
$(\mathrm{bool}^\perp \otimes \mathrm{bool})^\perp \otimes \mathrm{bool}$?

⋮

Conway games form a category:

* objects are Conway games
* an arrow from G to H is a strategy for $G^\perp \otimes H$

How do we compose such strategies?

Given $\sigma : G^\perp \otimes H$ and $\tau : H^\perp \otimes J$, define

$$\sigma \| \tau \triangleq \{ s \in (M_G + M_H + M_J)^* \mid s\restriction_{G,H} \in \sigma \wedge s\restriction_{HJ} \in \tau \}$$

& $\sigma ; \tau \triangleq \{ s\restriction_{G,J} \mid s \in \sigma \| \tau \}$

This gives a well-defined strategy for $G^\perp \otimes J$.

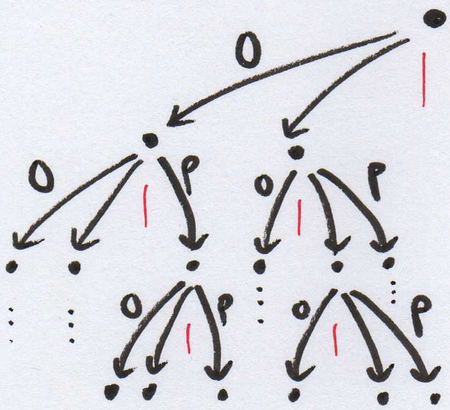
Identities are copycat strategies:

$$id : G^\perp \otimes G$$

$$id \triangleq \{ s \in P_{G^\perp \otimes G} \mid \forall s' \sqsubseteq^{even} s .\ s'\restriction_{G^\perp} = s'\restriction_G \}$$

Negative Conway Games

A Conway game is **negative** iff all opening moves belong to Opponent:



If G and H are both negative, then so is $G \otimes H$; but clearly $G^{\perp}$ isn't...

So, we define $G \multimap H$ by

* $M_{G \multimap H} \triangleq M_G + M_H$
* $\lambda_{G \multimap H} \triangleq [\overline{\lambda}_G, \lambda_H]$
* $P_{G \multimap H} \triangleq \{ s \in (M_G + M_H)^* \mid s\restriction_G \in P_G \wedge s\restriction_H \in P_H \}$

Since G is negative, all $s \in P_{G \multimap H}$ start in H.

The category of Conway games is compact closed whereas the category of negative Conway games is an SMCC :

$$\frac{(A \otimes B) \longrightarrow C}{A \longrightarrow (B \multimap C)}$$

It also has, unlike the whole category, <u>products</u> given by :

$$G \,\&\, H \triangleq \langle M_G + M_H,\ [\lambda_G, \lambda_H],\ ?\ \rangle$$

where

$$? = \{ s \in (M_G + M_H)^* \mid s \in P_G \vee s \in P_H \}$$

So $G \otimes H$ interleaves G and H whereas $G \,\&\, H$ plays in one or the other...

<u>Qu</u> What is $\langle \sigma, \tau \rangle : A \multimap (B \,\&\, C)$ given $\sigma : A \multimap B$ and $\tau : A \multimap C$?