Divisible load theory

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25 septembre et 2 octobre 2006

Overview

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- Problematic : to take into account the heterogeneity at the algorithmic level.

New platforms, new problems

Execution platforms : Distributed heterogeneous platforms (network of workstations, clusters, clusters of clusters, grids, etc.)

New sources of problems

- Heterogeneity of processors (computational power, memory, etc.)
- Heterogeneity of communications links.
- Irregularity of interconnection network.
- Non dedicated platforms.

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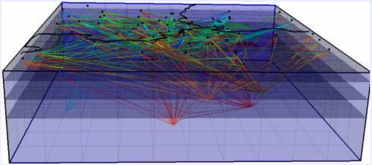
We need to adapt our algorithmic approaches and our scheduling strategies : new objectives, new models, etc.

An example of application : seismic tomography of the Earth

- Model of the inner structure of the Earth



- The model is validated by comparing the propagation time of a seismic wave in the model to the actual propagation time.
- Set of all seismic events of the year 1999 : 817101
- Original program written for a parallel computer :

```
if (rank = ROOT)
   raydata ← read n lines from data file ;
MPI_Scatter(raydata, n/P, ..., rbuff, ...,
            ROOT, MPI_COMM_WORLD) ;
compute_work(rbuff) ;
```

Applications covered by the divisible loads model

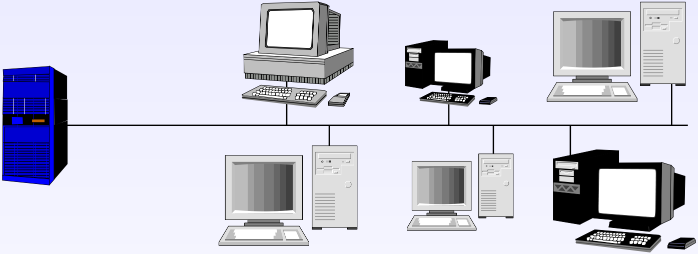
Applications made of a very (very) large number of fine grain computations.

Computation time proportional to the size of the data to be processed.

Independent computations : neither synchronizations nor communications.

Overview

Bus-like network



- The links between the master and the slaves all have the same characteristics.
- The slave have different computation power.

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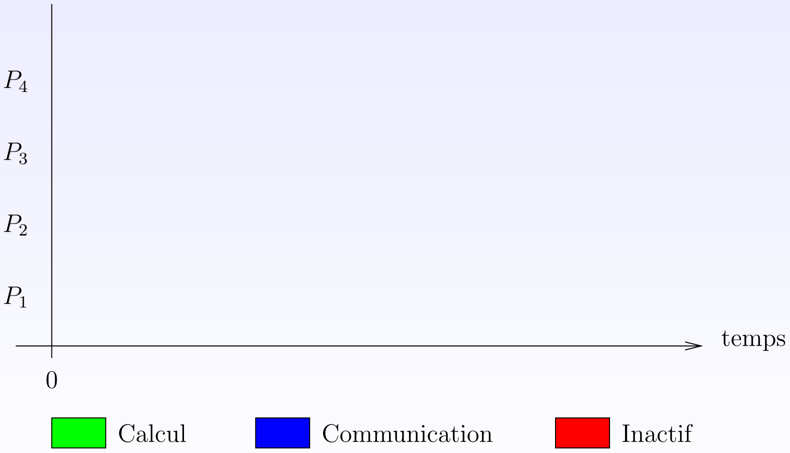
- Time needed to send a unit-message from P_1 to P_i : c.
  One-port bus : P_1 sends a *single* message at a time over the bus, all processors communicate at the same speed with the master.

Behavior of the master and of the slaves (illustration)



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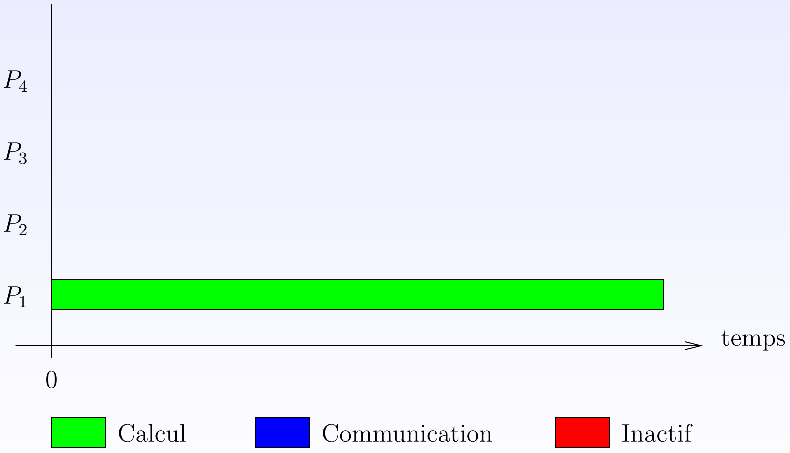

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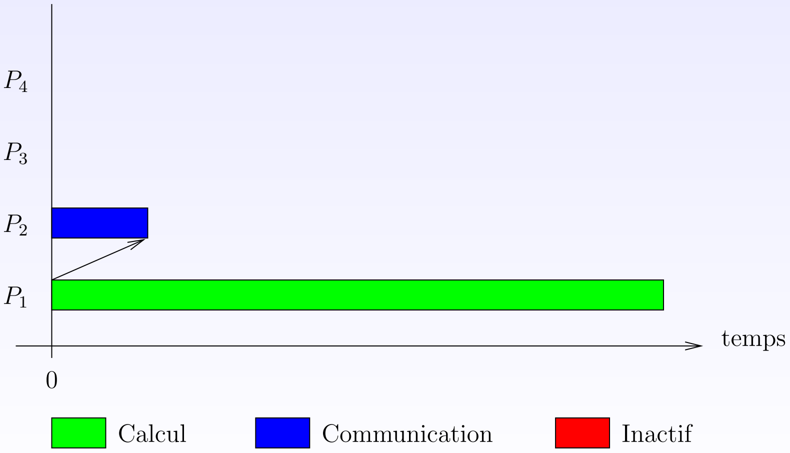

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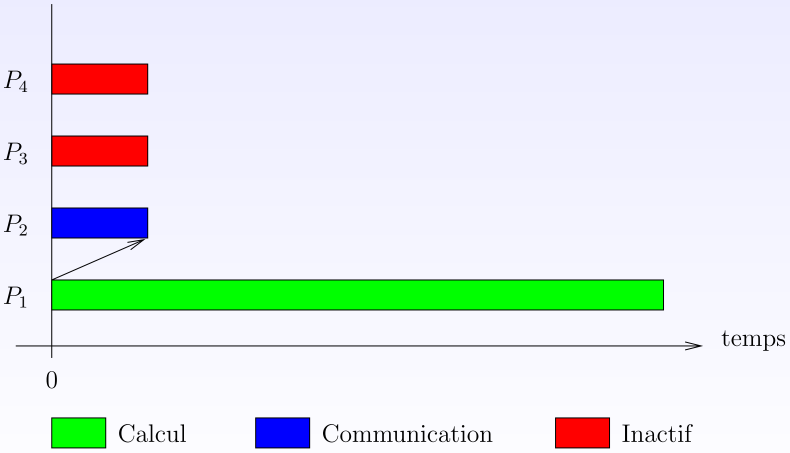

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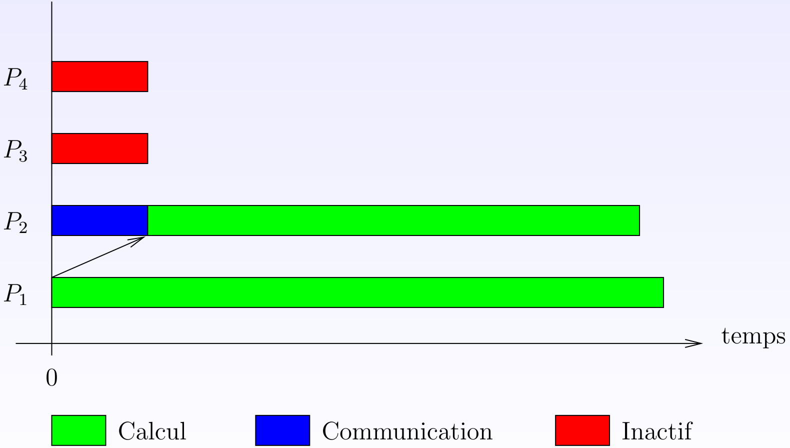

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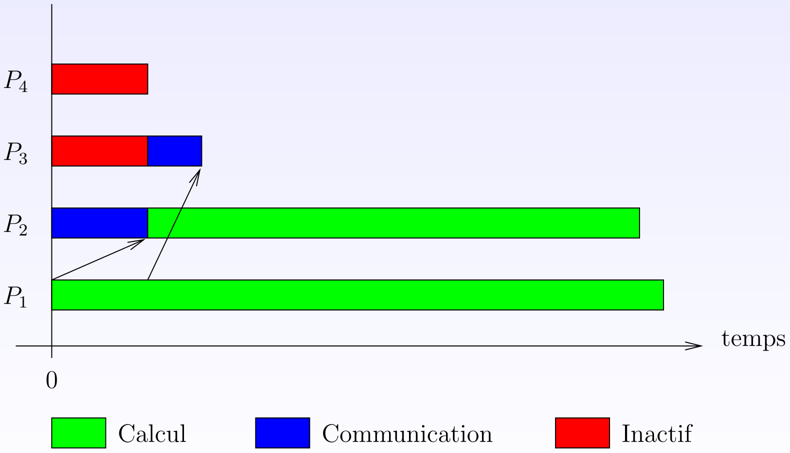

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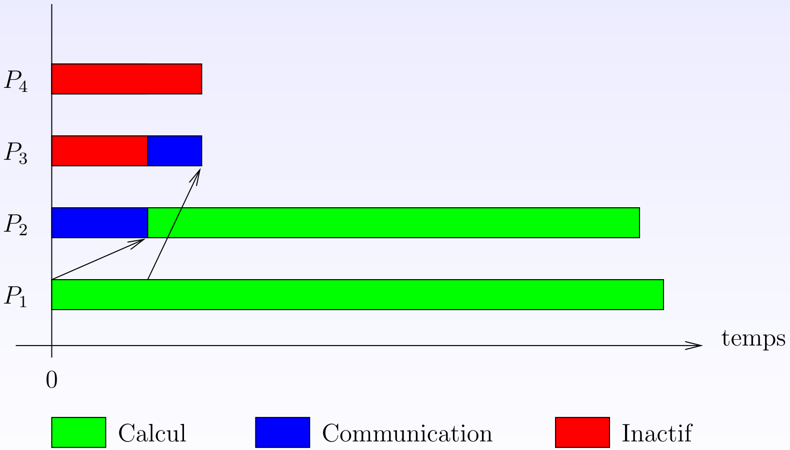

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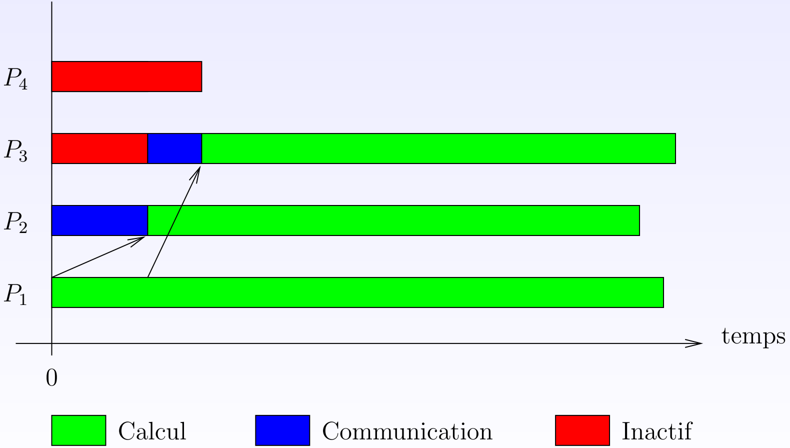

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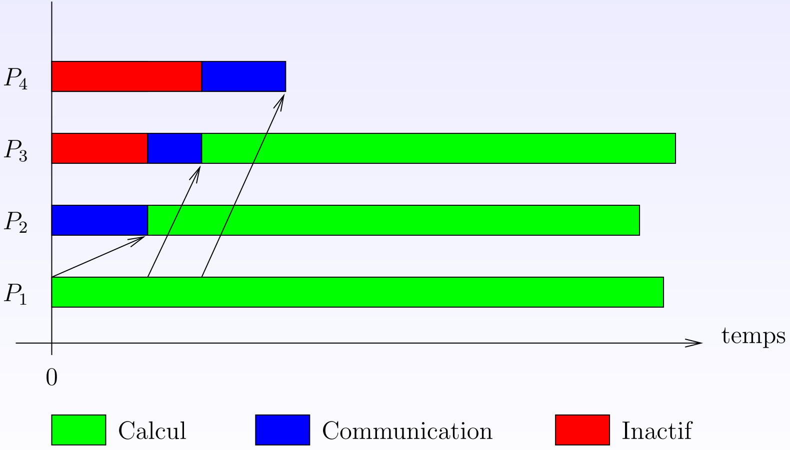

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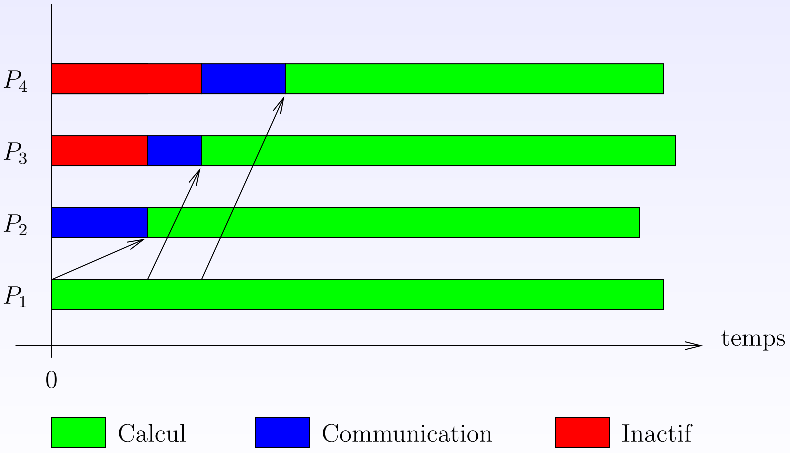

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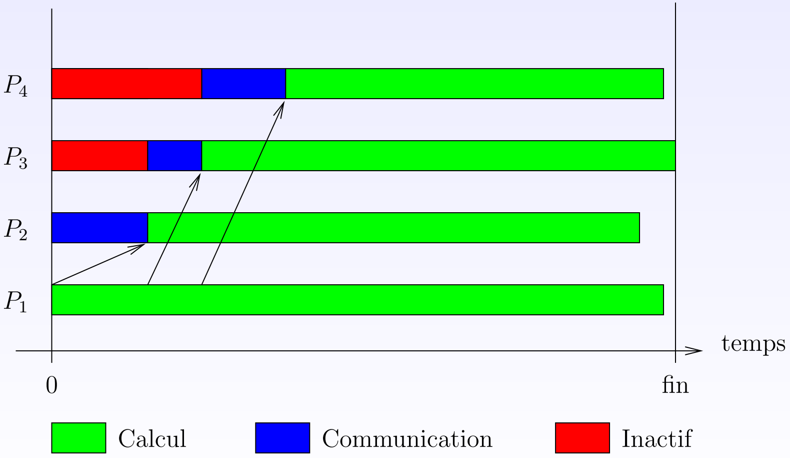

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- During this time the master processes its n_1 data.
- A slave does not start the processing of its data before it has received all of them.

Equations

- $P_1 : T_1 = n_1.w_1$

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- $P_i : T_i = \sum_{j=2}^{i} n_j.c + n_i.w_i$ for $i \geq 2$
- $P_i : T_i = \sum_{j=1}^{i} n_j.c_j + n_i.w_i$ for $i \geq 1$ with $c_1 = 0$ and $c_j = c$ otherwise.

Execution time

$$T = \max_{1 \leq i \leq p} \left(\sum_{j=1}^{i} n_j . c_j + n_i . w_i \right)$$

We look for a data distribution n_1, ..., n_p which minimizes T.

Execution time : rewriting

$$T = \max\left(n_1.c_1 + n_1.w_1, \max_{2\leq i\leq p}\left(\sum_{j=1}^{i} n_j.c_j + n_i.w_i\right)\right)$$

$$T = n_1.c_1 + \max\left(n_1.w_1, \max_{2\leq i\leq p}\left(\sum_{j=2}^{i} n_j.c_j + n_i.w_i\right)\right)$$

An optimal solution for the distribution of W_{total} data over p processors is obtained by distributing n_1 data to processor P_1 and then optimally distributing $W_{\text{total}} - n_1$ data over processors P_2 to P_p.

Algorithm

$\textit{solution}[0, p] \leftarrow \mathrm{cons}(0, NIL)$; $\textit{cost}[0, p] \leftarrow 0$

for $d \leftarrow 1$ **to** W_{total} **do**

  $\textit{solution}[d, p] \leftarrow \mathrm{cons}(d, NIL)$

  $\textit{cost}[d, p] \leftarrow d \cdot c_p + d \cdot w_p$

for $i \leftarrow p - 1$ **downto** 1 **do**

  $\textit{solution}[0, i] \leftarrow \mathrm{cons}(0, \textit{solution}[0, i + 1])$

  $\textit{cost}[0, i] \leftarrow 0$

  for $d \leftarrow 1$ **to** W_{total} **do**

    $(sol, min) \leftarrow (0, \textit{cost}[d, i + 1])$

    for $e \leftarrow 1$ **to** d **do**

      $m \leftarrow e \cdot c_i + \max(e \cdot w_i, \textit{cost}[d - e, i + 1])$

      if $m < min$ **then**

        $(sol, min) \leftarrow (e, m)$

    $\textit{solution}[d, i] \leftarrow \mathrm{cons}(sol, \textit{solution}[d - sol, i + 1])$

    $\textit{cost}[d, i] \leftarrow min$

return $(\textit{solution}[W_{\text{total}}, 1], \textit{cost}[W_{\text{total}}, 1])$

Complexity

- **Theorical complexity**

$$O(W_{\text{total}}^2 \cdot p)$$

- **Complexity in practice**
  If $W_{\text{total}} = 817101$ and $p = 16$, on a Pentium III running at 933 MHz : more than two days...
  (Optimized version ran in 6 minutes.)

Disadvantages

- Cost
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We do not need the solution to be so precise

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Equations

For processor P_i (with $c_1 = 0$ and $c_j = c$ otherwise) :

$$T_i = \sum_{j=1}^{i} \alpha_j W_{\text{total}}.c_j + \alpha_i W_{\text{total}}.w_i$$

$$T = \max_{1 \leq i \leq p} \left(\sum_{j=1}^{i} \alpha_j W_{\text{total}}.c_j + \alpha_i W_{\text{total}}.w_i \right)$$

We look for a data distribution α_1, ..., α_p which minimizes T.

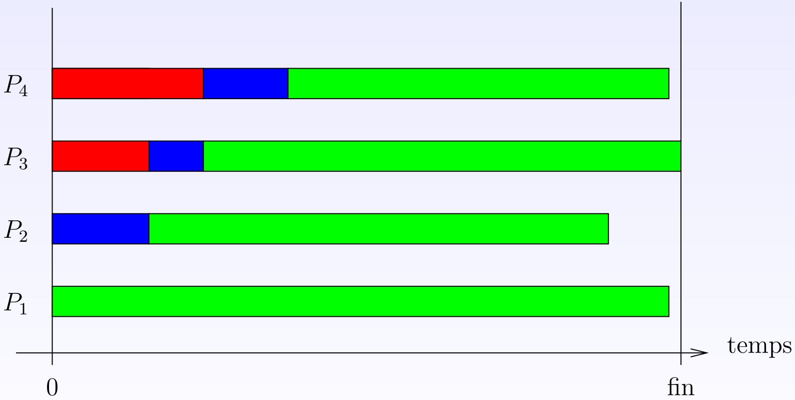
Properties of load-balancing

Lemma

In an optimal solution, all processors end their processing at the same time.

Demonstration of lemma 1

Two slaves i and $i+1$ with $T_i < T_{i+1}$.



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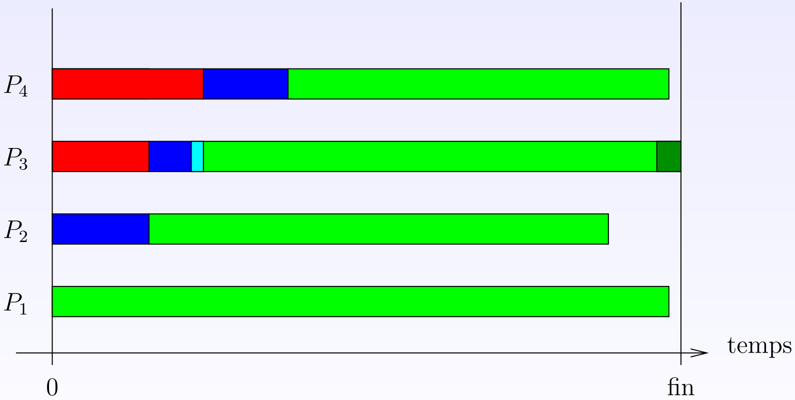

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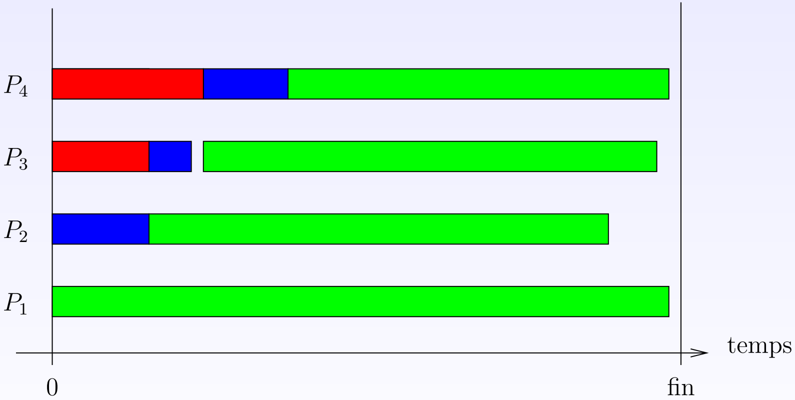

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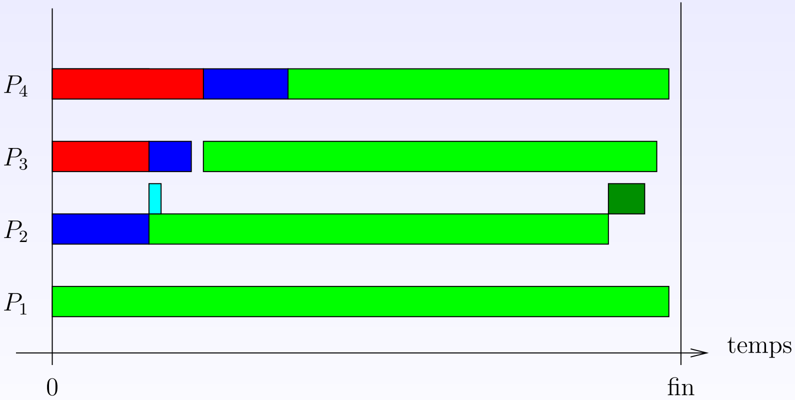

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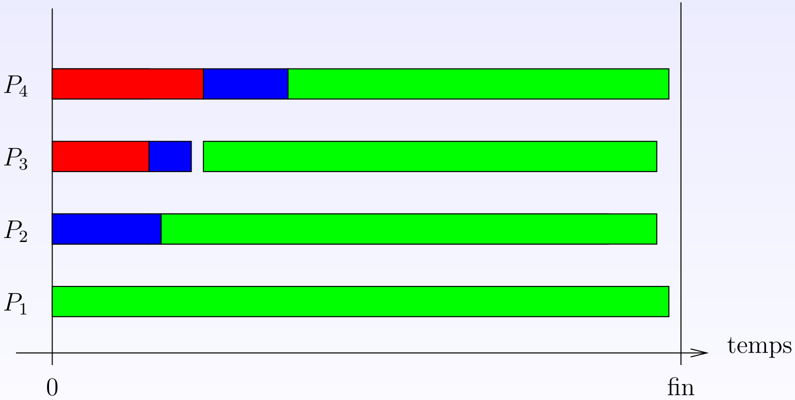

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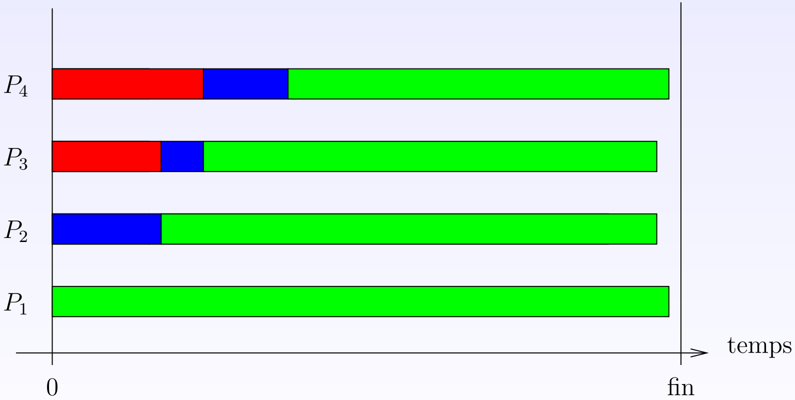

The communication time for the following processors is unchanged.

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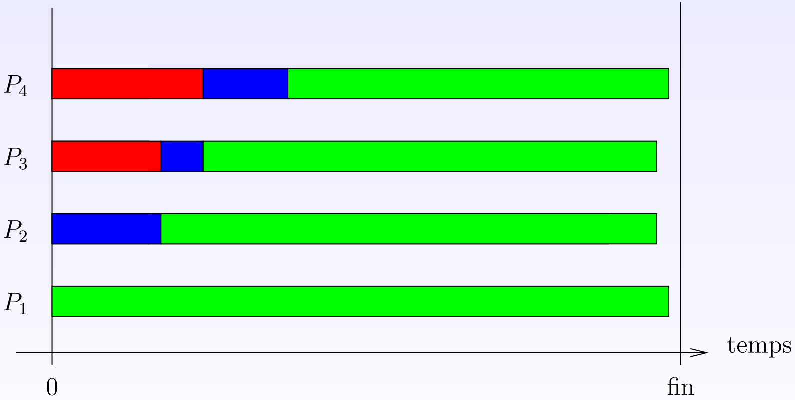

We end up with a better solution !

Demonstration of lemma 1 (continuation and conclusion)

- Ideal : $T'_i = T'_{i+1}$.
  We choose ϵ such that :

$$(\alpha_i + \epsilon)W_{\text{total}}(c + w_i) = (\alpha_i + \epsilon)W_{\text{total}}c + (\alpha_{i+1} - \epsilon)W_{\text{total}}(c + w_{i+1})$$

- The master stops before the slaves : absurde.
- The master stops after the slaves : we decrease P_1 by ϵ.

Property for the selection of ressources

Lemma

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Demonstration : this is just a corollary of lemma 1...

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$$\alpha_1 \left(1 + \frac{w_1}{c + w_2} + \ldots + \prod_{k=2}^{j} \frac{w_{k-1}}{c + w_k} + \ldots \right) = 1$$

Impact of the order of communications

How important is the influence of the ordering of the processor on the solution ?

?

No impact of the order of the communications

Volume processed by processors P_i and P_{i+1} during a time T.

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Thus $\alpha_{i+1} = \frac{1}{c+w_{i+1}}(\frac{T}{W_{\text{total}}} - \alpha_i c) = \frac{w_i}{(c+w_i)(c+w_{i+1})}\frac{T}{W_{\text{total}}}$.

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Processors P_i **and** P_{i+1} **:**

$$\alpha_i + \alpha_{i+1} = \frac{c+w_i+w_{i+1}}{(c+w_i)(c+w_{i+1})}$$

Choice of the master processor

We compare processors P_1 and P_2.

Choice of the master processor

We compare processors P_1 and P_2.

Processor P_1 **:** $\alpha_1 w_1 W_{\text{total}} = T$. Then, $\alpha_1 = \frac{1}{w_1}\frac{T}{W_{\text{total}}}$.

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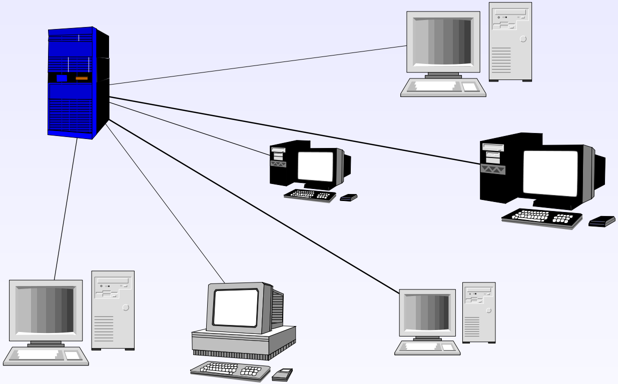
Minimal when $w_1 < w_2$.
Master = the most powerfull processor (for computations).

Conclusion

- Closed-form expressions for the execution time and the distribution of data.
- Choice of the master.
- The ordering of the processors has no impact.
- All processors take part in the work.

Overview

Star-like network



- The links between the master and the slaves have *different* characteristics.
- The slaves have different computational power.

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Processors must be served by decreasing bandwidths.

Ressource selection

Lemma

In an optimal solution, all processors work.

Demonstration of lemma 3

We take an optimal solution. Let P_k be a processor which does not receive any work : we put it last in the processor ordering and we give it a fraction α_k such that $\alpha_k(c_k + w_k)W_{\text{total}}$ is equal to the processing time of the last processor which received some work.

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Why should we put this processor last ?

Load-balancing property

Lemma

In an optimal solution, all processors end at the same time.

Demonstration of lemma 4

- Most existing proofs are false.

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$$\begin{cases} & \sum_{i=1}^{n} \alpha_i = 1 \\ \forall i, & \alpha_i \geq 0 \\ \forall i, & \sum_{k=1}^{i} \alpha_k c_k + \alpha_i w_i \leq T \end{cases}$$

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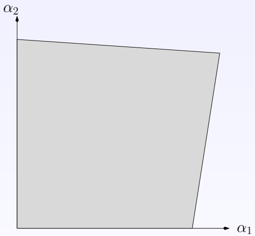

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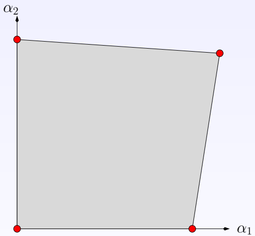

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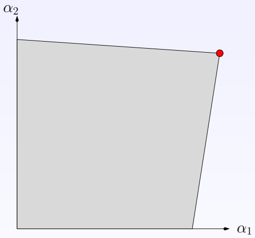

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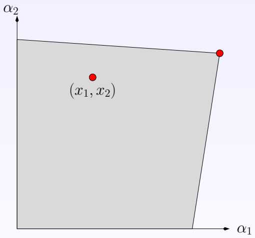

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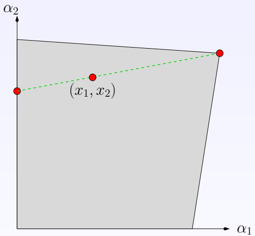

Conclusion

- The processors must be ordered by decreasing bandwidths
- All processors are working
- All processors end their work at the same time
- Formulas for the execution time and the distribution of data

Overview

With return messages

- Once it has finished processing its share of the total load, a slave sends back a result to the master.

- Problems to be solved :
  - Resource selection.
  - Defining an order for sending the data to the slaves.
  - Defining an order for receiving the data from the slaves.
  - Defining the amount of work each processor has to process.

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We need to anticipate, when building a solution, the possibility of idle times.

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  - Possibility to slow down a processor (to avoid idle times).
  - In practice : communication capabilities are not heterogeneous.
  - All FIFO distributions are equivalent and are better than any other solution (proof made by exchange).

Linear program for a given scenario (1)

A scenario is described by :

- which processor is given work to ;
- in which order the communications take place (sending of the data and gathering of the results).

With a given scenario, one can suppose that :

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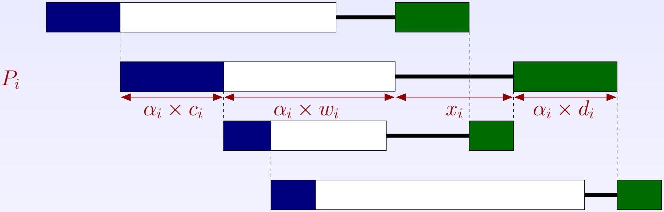

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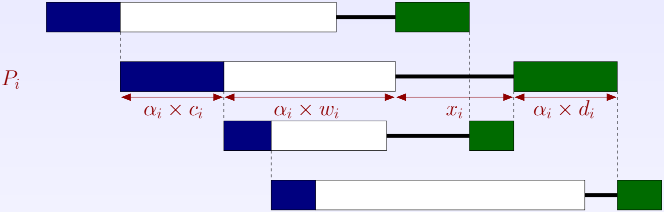

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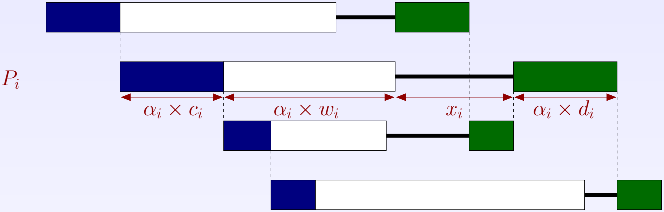

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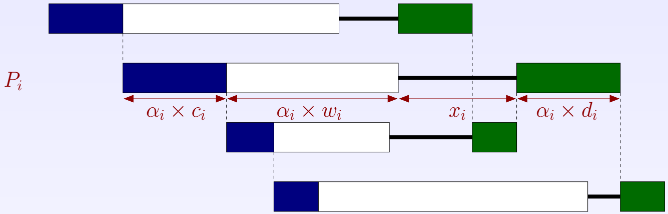

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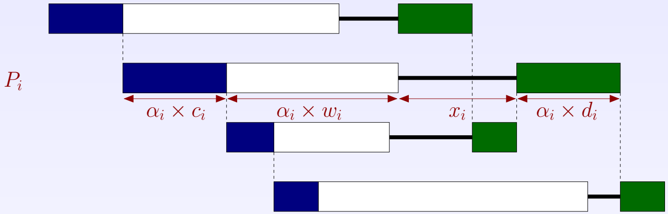

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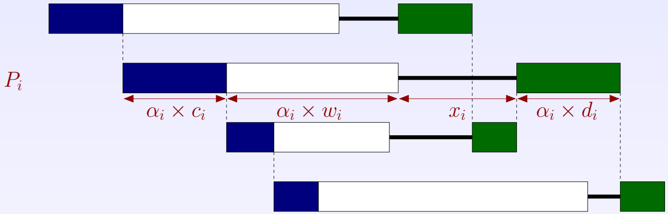

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- its idle time is : $x_i = t_i^{\text{back}} - t_i^{\text{term}} \geq 0$

Linear program for a given scenario (3)

For a given value of T, we obtain the linear program :

$$\textsc{Maximize } \sum_i \alpha_i, \textsc{ under the constraints} \\ \begin{cases} \alpha_i \geq 0 \\ t_i^{\text{back}} - t_i^{\text{term}} \geq 0 \end{cases} \quad (1)$$

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For a given amount of work $\sum_i \alpha_i = W$:

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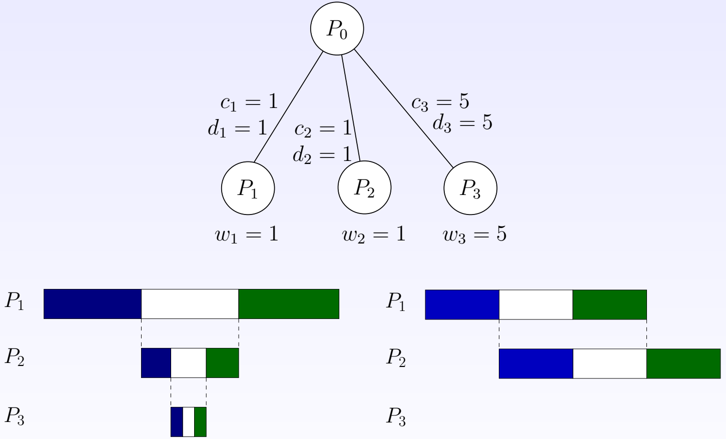
- Minimal time, an ordering and the resource selection being given.

Linear program for a given scenario (4)

One cannot test all possible configurations

- Even if we decide that the order of return messages should be the same than the order of data distribution messages (FIFO), there still is an exponential number of scenarios to be tested.

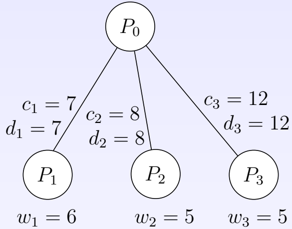
All processors do not always participate

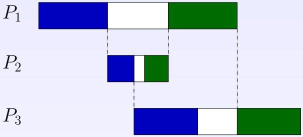


LIFO, throughput $\rho = 61/135$ (best schedule with 3 processors)

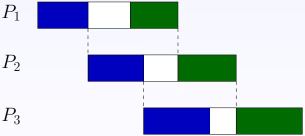
FIFO with 2 processors, optimal throughput$\rho = 1/2$

The optimal schedule may be neither LIFO nor FIFO

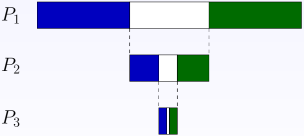




Optimal schedule
($\rho = 38/499 \approx 0.076$)



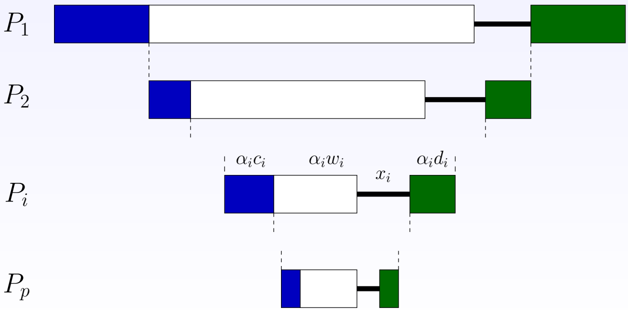
Best FIFO schedule
($\rho = 47/632 \approx 0.074$)



Best LIFO schedule
($\rho = 43/580 \approx 0.074$)

LIFO strategies (1)

- LIFO = Last In First Out
- The processor which receives its data first is the last to send its results back.
- The order of the return messages is the inverse of the order in which data are sent.



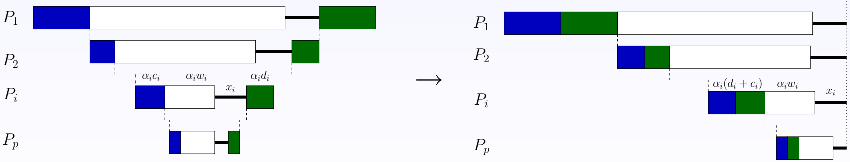
LIFO strategies (2)

Theorem

In the best LIFO solution :

- *All processors work*
- *The data are sent by increasing values of* $c_i + d_i$
- *There is no idle time, i.e.* $x_i = 0$ *for each* i*.*

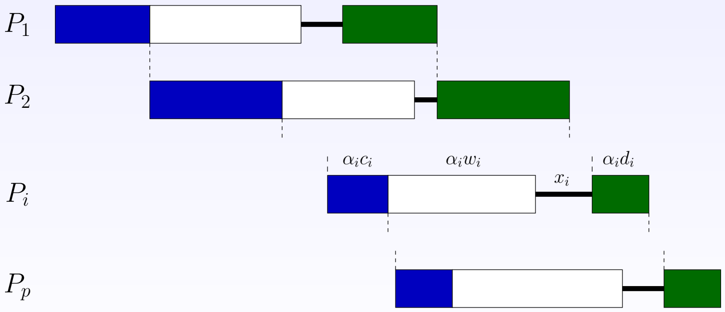
Demonstration : We change the platform : $c_i \leftarrow c_i + d_i$ and $d_i \leftarrow 0$



$\Rightarrow$ reduction to a classical problem without return messages.

FIFO strategies (1)

- FIFO = First In First Out
- The order the data are sent is the same than the order the return messages are sent.



We only consider the case $d_i = z \times c_i$ $(z < 1)$

FIFO strategies (2)

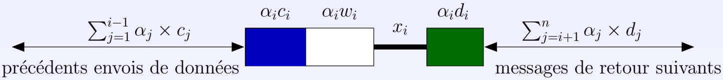
Theorem

In the best FIFO solution :

- *The data are sent by increasing values of :* $c_i + d_i$
- *The set of all working processors are made of the first* q *processors under this order ;* q *can be computed in linear time.*
- *There is no idle time, i.e.* $x_i = 0$ *for each* i*.*

FIFO strategies (3)

We consider i in the schedule :



$$\sum_{j=1}^{i} \alpha_i \times c_i + \alpha_i \times + w_i \sum_{j=i}^{n} \alpha_i \times d_i + x_i = T$$

We thus have : $A\alpha + x = T\mathbb{1}$, where :

$$A = \begin{pmatrix} c_1 + w_1 + d_1 & d_2 & d_3 & \dots & d_k \\ c_1 & c_2 + w_2 + d_2 & d_3 & \dots & d_k \\ \vdots & c_2 & c_3 + w_3 + d_3 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & d_k \\ c_1 & c_2 & c_3 & \dots & c_k + w_k + d_k \end{pmatrix}$$

FIFO strategies (4)

We can write $A = L + \mathbb{1}d^T$, with :

$$L = \begin{pmatrix} c_1 + w_1 & 0 & 0 & \dots & 0 \\ c_1 - d_1 & c_2 + w_2 & 0 & \dots & 0 \\ \vdots & c_2 - d_2 & c_3 + w_3 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ c_1 - d_1 & c_2 - d_2 & c_3 - d_3 & \dots & c_k + w_k \end{pmatrix} \text{ and } d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_k \end{pmatrix}$$

The matrix $\mathbb{1}d^t$ is a matrix of rank one, we can thus use Sherman-Morrison's formula to compute the inverse of A :

$$A^{-1} = (L + \mathbb{1}d^t)^{-1} = L^{-1} - \frac{L^{-1}\mathbb{1}d^t L^{-1}}{1 + d^t L^{-1}\mathbb{1}}$$

FIFO strategies (5)

With the formula which gives A^{-1}, one can :

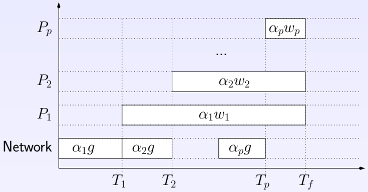
- show that for each processor P_i, either $\alpha_i = 0$ (the processor does not work) or $x_i = 0$ (no idle time) ;
- define analytically the throughput $\rho(T) = \sum_i \alpha_i$;
- show that the throughput is best when $c_1 \leq c_2 \leq c_3 \ldots \leq c_n$;
- show that the throughput is best when the only working processors are the one satisfying $d_i \leq \frac{1}{\rho_{\text{opt}}}$

FIFO strategies — special cases

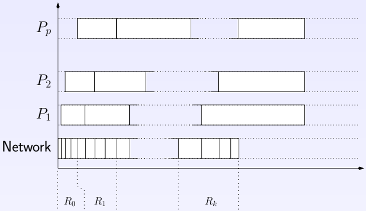
- So far, we have supposed that $d_i = z \times c_i$, with $z < 1$.
- If $z > 1$, symmetrical solution (the data are sent by decreasing values of $d_i + c_i$, the first q processors are selected under this order).
- $z = 1 \Rightarrow$ the order has no impact (but all processors do not always work).

Overview

One round vs. multi-round

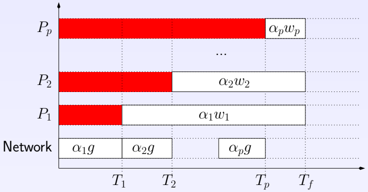


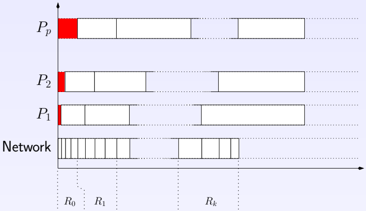
One round



Multi-round

One round vs. multi-round





One round

$\rightsquigarrow$ long idle-times

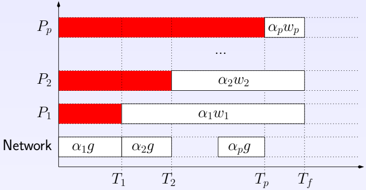
Multi-round

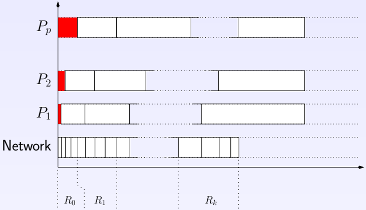
Efficient when W_{total} large

Intuition : start with small rounds, then increase chunks.

Problems :

One round vs. multi-round





One round

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Multi-round

Efficient when W_{total} large

Intuition : start with small rounds, then increase chunks.

Problems :

- linear communication model leads to absurd solution
- resource selection
- number of rounds
- size of each round

Notations

- A set P_1, ..., P_p of processors
- P_1 is the master processor : initially, it holds all the data.
- The overall amount of work : W_{total}.
- Processor P_i receives an amount of work $\alpha_i W_{\text{total}}$ with $\sum_i n_i = W_{\text{total}}$ with $\alpha_i W_{\text{total}} \in \mathbb{Q}$ and $\sum_i \alpha_i = 1$.
  Length of a unit-size work on processor P_i : w_i.
  Computation time on P_i : $n_i w_i$.
- **Time needed to send a message of size α_i P_1 to P_i : $L_i + c_i \times \alpha_i$.**
  One-port model : P_1 sends and receives a *single* message at a time.

Complexity

Definition (One round, $\forall i, c_i = 0$)

Given W_{total}, p workers, $(P_i)_{1 \le i \le p}$, $(L_i)_{1 \le i \le p}$, and a rational number $T \ge 0$, and assuming that bandwidths are infinite, is it possible to compute all W_{total} load units within T time units ?

Theorem

The problem with one-round and infinite bandwidths is NP-complete.

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What is the complexity of the general problem with finite bandwidths and several rounds ?

The general problem is NP-hard, but does not appear to be in NP (no polynomial bound on the number of activations).

Fixed activation sequence

Hypotheses

1. Number of activations : N_{act} ;
2. Whether P_i is **the** processor used during activation j : $\chi_i^{(j)}$

MINIMIZE T, UNDER THE CONSTRAINTS

$$\begin{cases} \sum_{j=1}^{N_{\text{act}}} \sum_{i=1}^{p} \chi_i^{(j)} \alpha_i^{(j)} = W_{\text{total}} \\ \forall k \leq N_{\text{act}}, \forall l : \left(\sum_{j=1}^{k} \sum_{i=1}^{p} \chi_i^{(j)} (L_i + \alpha_i^{(j)} c_i) \right) + \sum_{j=k}^{N_{\text{act}}} \chi_l^{(j)} \alpha_l^{(j)} w_l \leq T \\ \forall i, j : \alpha_i^{(j)} \geq 0 \end{cases} \quad (3)$$

Can be solved in polynomial time.

Fixed number of activations

Minimize T, under the constraints

$$\begin{cases} \sum_{j=1}^{N_{\text{act}}} \sum_{i=1}^{p} \chi_i^{(j)} \alpha_i^{(j)} = W_{\text{total}} \\ \forall k \leq N_{\text{act}}, \forall l : \left(\sum_{j=1}^{k} \sum_{i=1}^{p} \chi_i^{(j)} (L_i + \alpha_i^{(j)} c_i) \right) + \sum_{j=k}^{N_{\text{act}}} \chi_l^{(j)} \alpha_l^{(j)} w_l \leq T \\ \forall k \leq N_{\text{act}} : \sum_{i=1}^{p} \chi_i^{(k)} \leq 1 \\ \forall i, j : \chi_i^{(j)} \in \{0, 1\} \\ \forall i, j : \alpha_i^{(j)} \geq 0 \end{cases} \tag{4}$$

Exact but exponential

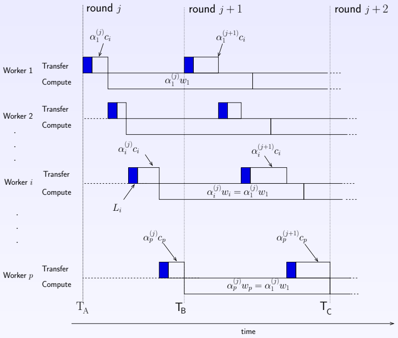
Can lead to branch-and-bound algorithms

Uniform multi-round

In a round : all workers have same computation time

Geometrical increase of rounds size

No idle time in communications :



$$\alpha_i^{(j)} w_i = \sum_{k=1}^{p} (L_k + \alpha_k^{(j+1)} c_k).$$

Heuristic processor selection : by decreasing bandwidths

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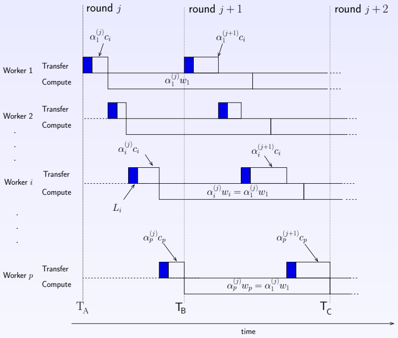

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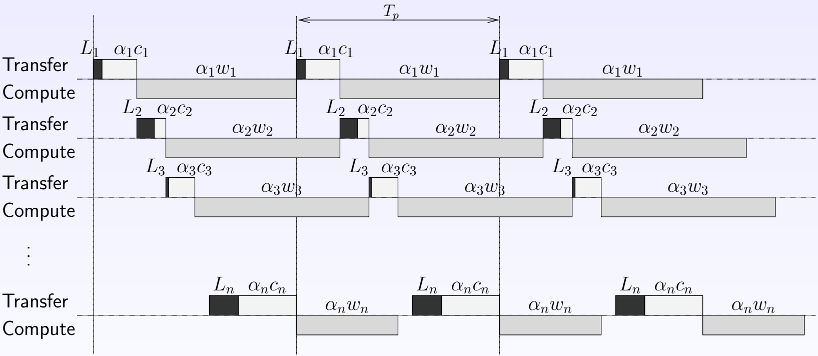
Heuristic processor selection : by decreasing bandwidths

No guarantee...

Periodic schedule



How to choose T_p ? Which resources to select ?

With no overlap (1/4)

Equations

- Divide total execution time T into k periods of duration T_p.

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- No overlap :

$$\forall i \in \mathcal{I}, \qquad L_i + \alpha_i (c_i + w_i) \leq T_p.$$

With no overlap (2/4)

Normalization

- β_i average number of tasks processed by P_i during one time unit.

With no overlap (2/4)

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- Linear program :

$$\begin{array}{l}\textsc{Maximize} \sum_{i=1}^{p} \beta_i \\ \left\{ \begin{array}{ll} \forall i \in \mathcal{I}, & \beta_i(c_i + w_i) \leq 1 - \frac{L_i}{T_p} \\ \sum_{i \in \mathcal{I}} \beta_i c_i \leq 1 - \frac{\sum_{i \in \mathcal{I}} L_i}{T_p} \end{array} \right. \end{array} .$$

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Relaxed version

$$\begin{array}{l}\textsc{Maximize} \sum_{i=1}^{p} x_i \\ \left\{\begin{array}{ll} \forall 1 \leq i \leq p, & x_i(c_i + w_i) \leq 1 - \frac{L_i}{T_p} \\ \sum_{i=1}^{p} x_i c_i \leq 1 - \frac{\sum_{i=1}^{p} L_i}{T_p} \end{array}\right.\end{array}$$

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With no overlap (3/4)

Bandwidth-centric solution

- Sort : $c_1 \leq c_2 \leq \ldots \leq c_p$.
- Let q be the largest index so that $\sum_{i=1}^{q} \frac{c_i}{c_i+w_i} \leq 1$.
- If $q < p$, $\epsilon = 1 - \sum_{i=1}^{q} \frac{c_i}{c_i+w_i}$.
- Optimal solution to relaxed program :

$$\forall 1 \leq i \leq q, \quad x_i = \frac{1 - \frac{\sum_{i=1}^{p} L_i}{T_p}}{c_i + w_i}$$

and (if $q < p$) :

$$x_{q+1} = \left(1 - \frac{\sum_{i=1}^{p} L_i}{T_p}\right)\left(\frac{\epsilon}{c_{q+1}}\right),$$

and $x_{q+2} = x_{q+3} = \ldots = x_p = 0$.

With no overlap (4/4)

Asymptotic optimality

- Let $T_p = \sqrt{T^*_{\max}}$ and $\alpha_i = x_i T_p$ for all i.

With no overlap (4/4)

Asymptotic optimality

- Let $T_p = \sqrt{T^*_{\max}}$ and $\alpha_i = x_i T_p$ for all i.
- Then $T \leq T^*_{\max} + O(\sqrt{T^*_{\max}})$.

With no overlap (4/4)

Asymptotic optimality

- Let $T_p = \sqrt{T^*_{\max}}$ and $\alpha_i = x_i T_p$ for all i.
- Then $T \leq T^*_{\max} + O(\sqrt{T^*_{\max}})$.
- Closed-form expressions for resource selection and task assignment provided by the algorithm.

With overlap

Key points

- Still sort resources according to the c_i.
- Greedily select resources until the sum of the ratios $\frac{c_i}{w_i}$ $\left(\text{instead of } \frac{c_i}{c_i+w_i}\right)$ exceeds 1.

Overview

Que retenir de tout ça ?

- Idée de base simple : une solution approchée est amplement suffisante.

- Les temps de communication jouent un plus grand rôle que les vitesses de calcul.