DEA Programmation
Preuves constructives
Corrigé

(1.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall y\ (P\ y) \vdash \forall y\ (P\ y)}\ \text{axiome}}{\forall y\ (P\ y) \vdash (P\ x)}\ \forall\text{-élim}}{\vdash ((\forall y\ (P\ y)) \Rightarrow (P\ x))}\ \Rightarrow\text{-intro}}{\vdash \forall x\ ((\forall y\ (P\ y)) \Rightarrow (P\ x))}\ \forall\text{-intro}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x\ ((P\ x) \Rightarrow (P\ (S\ x)))}\ \text{axiome}}{\Gamma \vdash (P\ (S\ 0)) \Rightarrow (P\ (S\ (S\ 0)))}\ \forall\text{-élim} \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x\ ((P\ x) \Rightarrow (P\ (S\ x)))}\ \text{axiome}}{\Gamma \vdash (P\ 0) \Rightarrow (P\ (S\ 0))}\ \forall\text{-élim} \quad \dfrac{}{\Gamma \vdash (P\ 0)}\ \text{axiome}}{\Gamma \vdash (P\ (S\ 0))}\ \Rightarrow\text{-élim}}{\Gamma \vdash (P\ (S\ (S\ 0)))}\ \Rightarrow\text{-élim}}{(P\ 0) \vdash \forall x\ ((P\ x) \Rightarrow (P\ (S\ x))) \Rightarrow (P\ (S\ (S\ 0)))}\ \Rightarrow\text{-intro}}{\vdash (P\ 0) \Rightarrow \forall x\ ((P\ x) \Rightarrow (P\ (S\ x))) \Rightarrow (P\ (S\ (S\ 0)))}\ \Rightarrow\text{-intro}$$

où $\Gamma = (P\ 0), \forall x\ ((P\ x) \Rightarrow (P\ (S\ x)))$

$$\dfrac{\dfrac{\dfrac{}{(P\ 0), (P\ 0) \vdash (P\ 0)}\ \text{axiome}}{(P\ 0) \vdash (P\ 0) \Rightarrow (P\ 0)}\ \Rightarrow\text{-intro}}{\vdash (P\ 0) \Rightarrow (P\ 0) \Rightarrow (P\ 0)}\ \Rightarrow\text{-intro}$$

(1.2)

$$\lambda x : \iota\ \lambda f : \Pi y : \iota\ (P\ y)\ (f\ x)$$

$$\lambda a : (P\ 0)\ \lambda f : \Pi x : \iota\ ((P\ x) \to (P\ (S\ x)))\ (f\ (S\ 0)\ (f\ 0\ a))$$

$$\lambda a : (P\ 0)\ \lambda b : (P\ 0)\ a$$

(2.1) Non.
(2.2) Simplement typé.
(2.3) $((\lambda x : B\ (f\ x\ h))\ (g\ h))$ dans $\Gamma = f : A \Rightarrow B \Rightarrow C, g : B \Rightarrow A, h : B$.
(2.4) $(f\ (g\ h)\ h)$.
(3.1)

$$t = (Rec\ \ < 0, i(refl\ 0) >\ \ \lambda z : nat\ \lambda f : P\ \ < z, j(refl\ (S\ z)) >)$$

où $P = \exists y\ ((z = 0) \vee (z = S(y)))$.
(3.2) $\exists y\ (((S\ (S\ (S\ (S\ 0)))) = 0) \vee ((S\ (S\ (S\ (S\ 0)))) = S(y)))$.
(3.3) $(S\ (S\ (S\ 0)))$.
(3.4) $nat \to nat$.
(3.5) $(S\ (S\ (S\ 0)))$.
(3.6) Avec t de la question 1 et

$$u = (Rec\ \ < S(0), i(refl\ 0) >\ \ \lambda z : nat\ \lambda f : P\ \ < z, j(refl\ (S\ z)) >)$$

on a $(\pi_1\ (t\ 0)) = 0$ et $(\pi_1\ (u\ 0)) = (S\ 0)$.