Examen – Probabilités

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1 Easy stuff

1.1 Casino

An individual uses the following gambling system. He bets 1 dollar that the roulette wheel will come up red. If he wins, he quits. If he loses then he makes the same bet a second time : only this time he bets 2 ; and then regardless of the outcome he quits. Assuming that he has a probability of 1/2 of winning each bet, what is the probability that he goes home a winner ? Why is this system not used by everyone ? avec un gain ? pourquoi tout le monde ne fait-il pas comme lui ?

1.2 Markov !

Consider a Markov chain $(X_0, X_1, \dots)$ with transition matrix P. Let $Y_n = X_{2n}$. If $(Y_0, Y_1, \dots)$ a Markov chain ? if yes, what is its transition matrix ?

1.3 Playing chess

1. Show that an irreducible Markov chain with a state i s.t. $P_{i,i} > 0$ is aperiodic
2. Consider a chessboard with a lone king making random moves (at each move he picks one of the possible squares to move to, uniformly at random). Is the corresponding Markov chain irreducible and/or aperiodic ?
3. Same question, except for a bishop
4. Same question, except for a knight

1.4 Markov ?

Consider the Markov chain with three states $\{1, 2, 3\}$ and transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ For all n, let $Y_n = \begin{cases} 0 \text{ si } X_n = 1 \\ 1 \text{ otherwise} \end{cases}$ Show that $(Y_0, Y_1, \dots)$ is not a Markov chain.

2 Serious stuff

2.1 Gala

n persons go to the ENS gala and leave their coats at the cloakroom. Unfortunately, the organization is not good and upon leaving everyone gets back one of the n coats randomly.

1. We have a *match* when a person gets his/her own coat back. Compute (for instance, by induction on n) the probability to have 0 match. Compute the probability to have exactly k matches.
2. Let X_k the random variable whose value is 1 if the k-th person gets his/her own coat back and 0 otherwise. Let also $S_n = X_1 + \cdots + X_n$ (representing the number of matches). Compute $E(S_n)$ and $Var(S_n)$
3. Show that $Pr(S_n \geq 11) \leq 0.01$ for all $n \geq 11$

2.2 Successive draws

A coin has probability p of coming up head H and $q = 1 - p$ of coming up tail T.

1. We flip the coin several times, up to the first flip where we get HH (two consecutive heads - for instance N=2 if the first two flips are heads). Compute the expectation $E(N)$.
2. Now we look for the number of flips M needed to get the sequence THTTH. Compute the expectation $E(M)$. How that can be generalized ?

2.3 Stationary distribution of a rational Markov chain

Consider a regular Markov chain with n states and whose transition matrix P has rational coefficients. For each state i, let a_i be the least common multiple of the denominators of non-zero coefficients in the i-th row of P. Here is an algorithm due to Engle :

- Initialization : for all i, put a_i tokens on state i
- At each step :
  - for all i, if there are x_i tokens on state i, send $x_i p_{ij}$ of them to state j for all $j \neq i$
  - for all i, there remains a'_i tokens on state i. Add just enough tokens so as to get a multiple of a_i
- Iterate until the number of tokens in all states is unchanged

Example : with 3 states, let $P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$ Start with $x = a = (4, 2, 4)$ tokens. After the distribution (sending) phase, we get $a' = (5, 2, 3)$ which we complement into $x = (8, 2, 4)$ for the second step.

1. What is the fixed point reached in the example ?
2. Show that there are always enough tokens for the distributions
3. Show that the fixed point is a multiple of the stationary distribution vector π