CHAPTER 7

Computing rate-distortion functions

7.1. Introduction

The entropy of a given probability distribution is easy to compute. The rate-distortion function of a distribution with respect to a distortion measure is just defined as the infimum value of a minimization problem. The minimization problem fortunately enjoys nice features. It is stated in the following way:

$$\inf_{\mathbb{Q}_{Y|x}} \mathbb{E}_{\mathbb{Q}_x}\left[D\left(\mathbb{Q}_{Y|X} \| \mathbb{E}_{\mathbb{Q}_x}\left[\mathbb{Q}_{Y|X}\right]\right)\right] \quad \text{where } \mathbb{E}_{\mathbb{Q}_x}\left[\mathbb{E}_{\mathbb{Q}_{Y|X}}\left[\rho(X,Y)\right]\right] \leq \mathsf{D}\,,$$

or equivalently to

$$\inf_{\mathbb{Q}_{Y|x}} I\left(\mathbb{Q}_X;\mathbb{Q}_{Y|x}\right) \quad \text{where } \mathbb{E}_{\mathbb{Q}_x}\left[\mathbb{E}_{\mathbb{Q}_{Y|X}}\left[\rho(X,Y)\right]\right] \leq \mathsf{D}\,.$$

It should be pointed out that the set of possible values of $\mathbb{Q}_{Y|x}$ may be identified with a compact and convex subspace of $\mathbb{R}^{|\mathcal{Y}|\times|\mathcal{X}|}$. On the hand, we have seen that the functional $\mathbb{Q}_{Y|x} \mapsto I\left(\mathbb{Q}_X;\mathbb{Q}_{Y|x}\right)$ is convex. Hence the computation of the rate-distortion function is a special case of a classical problem: minimizing a convex function over a convex and compact domain.

In order to solve approximately this problem, we will proceed in two steps. We will first put the moment constraint in a more tractable form by resorting to the Lagrangian formalism. We will prove that the Lagragian satisfies a strong duality property. This duality will allow us to focus on a convex function defined on an easily described compact and convex domain. The minimization of this function will be carried out using an alternative minimization procedure.

7.2. Duality formulae

The Lagrangian is defined by

$$L(\mathbb{Q}_{Y|x};\lambda) = I\left(\mathbb{Q}_X;\mathbb{Q}_{Y|x}\right) + \lambda\left(\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}\left[\rho(X,Y)\right]\right] - \mathsf{D}\right)$$

for any conditional probability distribution $\mathbb{Q}_{Y|x}$and non-negative multiplier λ.

THEOREM 7.2.1. [STRONG DUALITY THEOREM FOR RATE DISTORTION FUNCTION] *Let $\mathbb{Q}_X$ denote any probability on $\mathcal{X}$, let the distortion ρ, and the Lagrangian be defined as above then*

$$R(D) = \sup_{\lambda \geq 0} \min_{\mathbb{Q}_{Y|x}} L(\mathbb{Q}_{Y|x}; \lambda) = \min_{\mathbb{Q}_{Y|x}} \sup_{\lambda \geq 0} L(\mathbb{Q}_{Y|x}; \lambda) .$$

REMARK 7.2.2. This duality formula could be derived by resorting to a general min-max Theorem such as Sion Min-Max Theorem (already used to compare minimax and maximin redundancy in universal coding). In the present context, ad hoc arguments allow a direct proof of the strong duality, moreover ad hoc arguments tell us something about the location of the saddlepoint.

The formulation $\mathsf{R}(D) = \sup_{\lambda \geq 0} \min_{\mathbb{Q}_{Y|x}} L(\mathbb{Q}_{Y|x}; \lambda)$ is computationally appealing, since for each $\lambda \geq 0$, we $\min_{\mathbb{Q}_{Y|x}} L(\mathbb{Q}_{Y|x}; \lambda)$ is now defined as an infimum over all conditional probabilities. As such it may seem easier to compute than the rate-distortion function itself. Second, if the infimum of

$$G(\lambda) \triangleq I\left(\mathbb{Q}_X; \mathbb{Q}_{Y|x}\right) + \lambda \left(\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}\left[\rho(X,Y)\right]\right]\right)$$

can be computed for every λ, then the supremum of $G(\lambda) - \lambda \mathsf{D}$, can be searched efficiently.

PROOF. Let us first check that

$$\mathsf{R}(D) = \min_{\mathbb{Q}_{Y|x}} \sup_{\lambda > 0} L(\mathbb{Q}_{Y|x}; \lambda)$$

If $\mathsf{R}(\mathsf{D}) = \infty$ (this may only occur if $\mathcal{X}$ is infinite), we are done.

Otherwise, if $\mathbb{Q}_{Y|x}$ is such that

$$\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}\left[\rho(X,Y)\right]\right] - \mathsf{D} \geq 0$$

the supremum with respect to $\lambda \geq 0$ is infinite. Assume that

$$\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}\left[\rho(X,Y)\right]\right] \leq \mathsf{D},$$

then $L(\mathbb{Q}_{Y|x}; \lambda) = I\left(\mathbb{Q}_X; \mathbb{Q}_{Y|x}\right)$, the minimum is achieved by picking the optimal joining.

Consider now

$$\sup_{\lambda \geq 0} \min_{\mathbb{Q}_{Y|x}} L(\mathbb{Q}_{Y|x}; \lambda).$$

Picking the optimal joining in the definition of $\mathsf{R}(\mathsf{D})$ we get that

$$\mathsf{R}(\mathsf{D}) \geq \sup_{\lambda \geq 0} \min_{\mathbb{Q}_{Y|x}} L(\mathbb{Q}_{Y|x}; \lambda).$$

Now by the monotonicity and convexity properties of the rate-distortion function there exists some $\lambda^* \geq 0$, such that for all $\mathsf{D}' > \mathsf{D}$,

$$\mathsf{R}(\mathsf{D}') \geq \mathsf{R}(\mathsf{D}) - \lambda^*(\mathsf{D}' - \mathsf{D}) ,$$

$-\lambda^*$ is a sub-gradient of $\mathsf{R}()$ at D. Consider now $L(\cdot, \lambda^*)$. Let $\mathbb{Q}_{Y|x}$ be such that $\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}[\rho(X,Y)]\right] = \mathsf{D}' > \mathsf{D}$. Then

$$L(\mathbb{Q}_{Y|x}; \lambda^*) \geq \mathsf{R}(\mathsf{D}') + \lambda^*(\mathsf{D}' - \mathsf{D}) \geq \mathsf{R}(\mathsf{D}) .$$

□

This proves that the Lagrangian has a saddlepoint which first component is the conditional distribution which witnesses the value of $\mathsf{R}(\mathsf{D})$ and which second component is the opposite of a sub-gradient of $\mathsf{R}()$at D.

In order to compute $G(\lambda)$, we will define a function $F(;)$ of two conditional probabilities in the following way:

$$F(\mathbb{Q}_{Y|x}; \mathbb{Q}'_{Y|x}) \triangleq \mathbb{E}_{\mathbb{Q}_X}\left[D\left(\mathbb{Q}_{Y|x} \| \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}'_{Y|x}\right]\right)\right] + \lambda \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}[\rho(X,Y)]\right] .$$

LEMMA 7.2.3. *For a fixed $\mathbb{Q}_{Y|x}$, the functional:*

$$\mathbb{Q}'_{Y|x} \mapsto F(\mathbb{Q}_{Y|x}; \mathbb{Q}'_{Y|x})$$

is minimized by $\mathbb{Q}'_{Y|x} = \mathbb{Q}_{Y|x}$,and the minimum equals

$$L(\mathbb{Q}_{Y|x}; \lambda) + \lambda D. \ .$$

The proof of this Lemma is just what we need to establish the remark at the end of th first section.

PROOF. It is enough to notice that:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_X}\left[D\left(\mathbb{Q}_{Y|x} \| \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}'_{Y|x}\right]\right)\right] + \lambda \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}[\rho(X,Y)]\right] \\
&\qquad = \mathbb{E}_{\mathbb{Q}_X}\left[D\left(\mathbb{Q}_{Y|x} \| \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}_{Y|x}\right]\right)\right] \\
&\qquad + D\left(\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}_{Y|x}\right] \| \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}'_{Y|x}\right]\right) + \lambda \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}[\rho(X,Y)]\right] .
\end{aligned}$$

On the right-hand-side, only the second summand depends on $\mathbb{Q}'_{Y|x}$ and it is non-negative, and positive if $\mathbb{Q}'_{Y|x} \neq \mathbb{Q}_{Y|x}$. □

EXERCISE 7.3. [ABOUT THE RATE-REDUNDANCY OF MIS-SPECIFIED RECONSTRUCTION PROBABILITIES] In the proof of the direct lossy source coding Theorem, we used the second marginal of the optimal distortion-compliant coupling as a tool to generate a small codebook with small average distortion. We might have generated a codebook using another probability $\mathbb{Q}'_Y$ to generate the codebook.

Check that it would have been enough to consider code-books of size $2^{n\mathsf{R}'}$ for

$$\log 2 \times \mathsf{R}' > D\left(\mathbb{Q} \| \mathbb{Q}_X \otimes \mathbb{Q}'_Y\right)$$

where $\mathbb{Q}$ is the coupling that witnesses the value of $\mathsf{R}(\mathsf{D}\ )$ and $\mathbb{Q}_X$is its first marginal, that is the distribution of the source. The difference

$$D\left(\mathbb{Q} \| \mathbb{Q}_X \otimes \mathbb{Q}'_Y\right) - \mathsf{R}(\mathsf{D})$$

is called the rate-redundancy of the quantizer (randomly) derived from $\mathbb{Q}'_Y$.

LEMMA 7.3.1. *For a fixed $\mathbb{Q}'_{Y|x}$, the functional:*

$$\mathbb{Q}_{Y|x} \mapsto F(\mathbb{Q}_{Y|x}; \mathbb{Q}'_{Y|x})$$

is minimized by

$$\widehat{\mathbb{Q}}_{Y|x}\{y \mid x\} \triangleq \frac{1}{Z(x)} \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}'_{Y|x}\{y \mid X\}\right] \times \exp\left(-\lambda \rho(x, y)\right)$$

where

$$Z(x) \triangleq \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}'_{Y|x}}\left[\exp\left(-\rho(x, Y)\right)\right]\right]$$

and the minimum equals

$$-\mathbb{E}_{\mathbb{Q}_X}\left[\log Z(X)\right] .$$

PROOF. Let $\mathbb{Q}_{Y|x}$ denote any conditional distribution, then

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_X}\left[D\left(\mathbb{Q}_{Y|x} \| \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}'_{Y|x}\right]\right)\right] + \lambda \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}\left[\rho(X, Y)\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}\left[\log \frac{\mathbb{Q}_{Y|x}\{Y \mid X\}}{\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}'_{Y|X'}\{Y \mid X'\}\right] \exp\left(-\lambda \rho(X, Y)\right)}\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}_{Y|x}}\left[\log \frac{Z(X) \times \mathbb{Q}_{Y|x}\{Y \mid X\}}{\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}'_{Y|X'}\{Y \mid X'\}\right] \exp\left(-\lambda \rho(X, Y)\right)}\right]\right] - \mathbb{E}_{\mathbb{Q}_X}\left[\log Z(X)\right] \\
&= \mathbb{E}_{\mathbb{Q}_X}\left[D\left(\mathbb{Q}_{Y|x} \| \widehat{\mathbb{Q}}_{Y|x}\right)\right] - \mathbb{E}_{\mathbb{Q}_X}\left[\log Z(X)\right] \\
&\geq -\mathbb{E}_{\mathbb{Q}_X}\left[\log Z(X)\right] .
\end{aligned}$$

In the last statement equality holds if and only if $\mathbb{Q}_{Y|x} = \widehat{\mathbb{Q}}_{Y|x}$. □

REMARK 7.3.2. Note that computing the rate-distortion function may also be viewed as computing the minimum relative entropy between two convex sets of probability distributions on $\mathcal{X} \times \mathcal{Y}$. The first set is constituted by those joint distributions which first marginal is $\mathbb{Q}_X$ and which satisfy $\mathbb{E}_{\mathbb{Q}}[\rho] \leq \mathsf{D}$. The second set is constituted by product distributions $\mathbb{Q}_X \otimes \mathbb{Q}'$ where $\mathbb{Q}'$ is any distribution over $\mathcal{Y}$.

LEMMA 7.3.3.

$$R(D) = -\min_{\mathbb{Q}'_X \in \mathfrak{M}_1(\mathcal{X})} \min_{\lambda > 0} \left\{ D\left(\mathbb{Q}_X \| \mathbb{Q}'_X\right) + \max_{y \in \mathcal{Y}} \log \left(\sum_{x \in \mathcal{X}} \mathbb{Q}'_X\{x\} \exp\left[\lambda(D - \rho(x,y))\right] \right) \right\}$$

7.4. The Blahut-Arimoto algorithm

In this Section, $\mathbb{Q}^*_{Y|x}$ denote the unique conditional distribution that minimizes $L(\cdot;\lambda)$,$\mathbb{Q}^*_Y$ denotes the second marginal of the joining by $\mathbb{Q}_X$ and $\mathbb{Q}^*_{Y|x}$.

The Blahut-Arimoto algorithm is actually an iterative algorithm for computing $\mathbb{Q}^*_{Y|x}$.

Define a sequence $\left(\mathbb{Q}^n_{Y|x}\right)_{n\in\mathbb{N}}$, by taking $\mathbb{Q}^1_{Y|x}$ such that every for all $y \in \mathcal{Y}$, $\mathbb{Q}^1_{Y|x}\{y\} > 0$. For any n,let $\mathbb{Q}^n_Y$ denote the Y marginal of the joining defined by $\mathbb{Q}_X$ and $\mathbb{Q}^n_{Y|x}$. The recurrence relation is defined by:

$$\mathbb{Q}^{n+1}_{Y|x}\{y \mid x\} = \frac{\exp\left(-\lambda\rho(x,y)\right)}{Z_n(x)} \mathbb{Q}^n_Y\{y\}$$

where $Z_n(x)$ are normalization constants

$$Z_n(x) = \sum_y \mathbb{Q}^n_Y\{y\} \exp\left(-\lambda\rho(x,y)\right) .$$

LEMMA 7.4.1. *Assume the sequence* $\left(\mathbb{Q}^n_{Y|x}\right)_{x\in\mathcal{X},n\in\mathbb{N}}$ *is defined as above, and* $\mathbb{Q}^*_{Y|x}$ *denote the conditional distribution that minimizes* $F(;)$, *then for all* n,

$$F\left(\mathbb{Q}^{n+1}_{Y|x};\mathbb{Q}^n_{Y|x}\right) - F\left(\mathbb{Q}^*_{Y|x};\mathbb{Q}^*_{Y|x}\right) \leq D\left(\mathbb{Q}^*_Y \| \mathbb{Q}^n_Y\right) - D\left(\mathbb{Q}^*_Y \| \mathbb{Q}^{n+1}_Y\right) .$$

PROOF. The first two equations follows from rearrangements of summations: $F\left(\mathbb{Q}^{n+1}_{Y|x};\mathbb{Q}^{n}_{Y|x}\right) - F\left(\mathbb{Q}^{*}_{Y|x};\mathbb{Q}^{*}_{Y|x}\right) =$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}^*_{Y|x}} \log \frac{\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}^*_{Y|x}\{Y|X'\}\right] e^{-\lambda\rho(X,Y)}}{\mathbb{Q}^*_{Y|x}\{Y|X\} Z_n(X)}\right] \\
&= \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}^*_{Y|x}} \log \frac{\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}^*_{Y|x}\{Y|X'\}\right]}{\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}^n_{Y|x}\{y \mid X\}\right]}\right] \\
&- \mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{E}_{\mathbb{Q}^*_{Y|x}} \log \frac{\mathbb{Q}^*_{Y|x}\{Y|X\}}{\mathbb{E}_{\mathbb{Q}_X}\left[\mathbb{Q}^{n+1}_{Y|x}\{Y \mid X\}\right]}\right] \\
&\leq D\left(\mathbb{Q}^*_Y \| \mathbb{Q}^n_Y\right) - D\left(\mathbb{Q}^*_Y \| \mathbb{Q}^{n+1}_Y\right) .
\end{aligned}
$$

where the last line comes from the convexity of relative entropy. □

THEOREM 7.4.2. *Assume the sequence* $\left(\mathbb{Q}^n_{Y|x}\right)_{x\in\mathcal{X},n\in\mathbb{N}}$ *is defined as above, and* $\mathbb{Q}^*_{Y|x}$ *denote the conditional distribution that minimizes* $F(;)$, *then for all* n :

$$
\begin{aligned}
0 \leq F\left(\mathbb{Q}^{n+1}_{Y|x};\mathbb{Q}^n_{Y|x}\right) - F\left(\mathbb{Q}^*_{Y|x};\mathbb{Q}^*_{Y|x}\right) \\
\leq -\max_{y\in\mathcal{Y}} \log\left(\sum_{x\in\mathcal{X}} \frac{\mathbb{Q}_X\{x\}}{Z_n(x)} \exp\left[-\lambda\rho(x,y)\right]\right)
\end{aligned}
$$

and $F\left(\mathbb{Q}^{n+1}_{Y|x};\mathbb{Q}^n_{Y|x}\right)$ *converges monotonically toward* $F\left(\mathbb{Q}^*_{Y|x};\mathbb{Q}^*_{Y|x}\right)$.

PROOF. The first inequality comes from the definition of $\mathbb{Q}^*_{Y|x}$.

The convergence follows from a telescoping sum argument. From Lemma above:

$$
\begin{aligned}
0 &\leq \sum_{n=1}^{N} F\left(\mathbb{Q}^{n+1}_{Y|x};\mathbb{Q}^n_{Y|x}\right) - F\left(\mathbb{Q}^*_{Y|x};\mathbb{Q}^*_{Y|x}\right) \\
&\leq D\left(\mathbb{Q}^*_Y \| \mathbb{Q}^1{}_Y\right) - D\left(\mathbb{Q}^*_Y \| \mathbb{Q}^{N+1}{}_Y\right) \\
&\leq D\left(\mathbb{Q}^*_Y \| \mathbb{Q}^1{}_Y\right) .
\end{aligned}
$$

This shows that $F\left(\mathbb{Q}^{n+1}_{Y|x};\mathbb{Q}^n_{Y|x}\right) - F\left(\mathbb{Q}^*_{Y|x};\mathbb{Q}^*_{Y|x}\right)$ is the generic term of an absolutely convergent series, hence it converges (monotonically) toward 0. This in turns imply that $\mathbb{Q}^n_{Y|x}$ converges toward $\mathbb{Q}^*_{Y|x}$. □