Game Theory and its Applications to Networks

Corinne Touati / Bruno Gaujal

Master ENS Lyon, Fall 2011

Course Overview

Part 1 (C. Touati) : Games, Solutions and Applications

Sept. 21	Introduction - Main Game Theory Concepts
Sept. 28	Special games - potential, super-additive, dynamical...
Oct. 5	Classical Sol. Algo. - Best Response and Fictitious Play
Oct. 19	Mechanism Design - Building a game
Nov. 2	Advanced concepts - Auctions and Coalitions

Part 2 (B. Gaujal) : Algorithmic Solutions from Evolutionary Games

Nov. 9	Evolutionary game theory and related dynamics
Nov. 16	From dynamics to algorithms
Nov. 23	Relationship with classical learning algorithms

Bibliography

- Roger Myerson, *"Game Theory: Analysis of Conflicts"*
- Guillermo Owen, *"Game Theory"*, 3rd edition
- Başar and Olsder, *"Dynamic Noncooperative Game Theory"*
- Walid Saad, *"Coalitional Game Theory for Distributed Cooperation in Next Generation Wireless Networks"* (Phd. Thesis)
- Nisan, Roughgarden, Tardos and Vazirani, *"Algorithmic Game Theory"*
- Weibull, *"Evolutionary Game Theory"*
- Borkar, *"Stochastic Approximation"*
- Michel Benaim, *"Dynamics of Stochastic Approximation Algorithms"* - Séminaire de probabilité (Strasbourg), tome 33, p1-68

Part I

Introduction: Main Concepts in Game Theory and a few applications

What is Game Theory and what is it for?

Definition (Roger Myerson, *"Game Theory, Analysis of Conflicts"*)

"Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare."

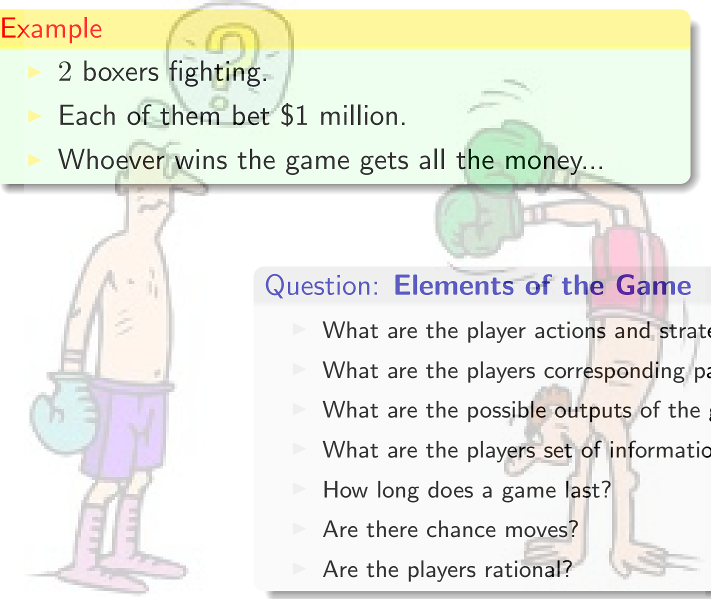
- Branch of optimization
- Multiple actors with different objectives
- Actors interact with each others

Game Theory and Nobel Prices

- Roger B. Myerson (2007, 1951) – eq. in dynamic games
- Leonid Hurwicz (2007, 1917-2008) – incentives
- Eric S. Maskin (2007, 1950) – mechanism design
- Robert J. Aumann (2005, 1930) – correlated equilibria
- Thomas C. Schelling (2005, 1921) – bargaining
- William Vickrey (1996, 1914-1996) – pricing
- Robert E. Lucas Jr. (1995, 1937) – rational expectations
- John C. Harsanyi (1994, 1920-2000) – Bayesian games, eq. selection
- John F. Nash Jr. (1994, 1928) – NE, NBS
- Reinhard Selten (1994, 1930) – Subgame perf. eq., bounded rationality
- Kenneth J. Arrow (1972, 1921) – Impossibility theorem
- Paul A. Samuelson (1970, 1915-2009) – thermodynamics to econ.

(Jorgen Weibull - Chairman 2004-2007)
(more info on http://lcm.csa.iisc.ernet.in/gametheory/nobel.html)

Example of Game



Example

- 2 boxers fighting.
- Each of them bet \$1 million.
- Whoever wins the game gets all the money...

Question: **Elements of the Game**

- What are the player actions and strategies?
- What are the players corresponding payoffs?
- What are the possible outputs of the game?
- What are the players set of information?
- How long does a game last?
- Are there chance moves?
- Are the players rational?

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case
2. Two Inspiring Examples
3. Optimality
4. Bargaining Concepts
5. Measuring the Inefficiency of a Policy
6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case
2. Two Inspiring Examples
3. Optimality
4. Bargaining Concepts
5. Measuring the Inefficiency of a Policy
6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

Pure Competition: Modeling

Definition:
Two Players, Zero-Sum Game.

- 2 players, finite number of actions
- Payoffs of players are opposite (and depend on both players' actions)

Modelization

- We call strategy a decision rule on the set of actions
- (Pure Strategy) Payoffs can be represented by a matrix A where
$$\left.\begin{array}{l}\text{Player } 1 \text{ chooses } i,\\ \text{Player } 2 \text{ chooses } j\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{player } 1 \text{ gets } a_{ij}\\ \text{player } 2 \text{ gets } -a_{ij}\end{array}\right.$$
- A solution point is such that

Pure Competition: Modeling

Definition:

Two Players, Zero-Sum Game.

- 2 players, finite number of actions
- Payoffs of players are opposite (and depend on both players' actions)

Modelization

- We call strategy a decision rule on the set of actions
- (Pure Strategy) Payoffs can be represented by a matrix A where

$$\left.\begin{array}{l}\text{Player } 1 \text{ chooses } i,\\ \text{Player } 2 \text{ chooses } j\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{player } 1 \text{ gets } a_{ij}\\ \text{player } 2 \text{ gets } -a_{ij}\end{array}\right.$$

- A solution point is such that no player has incentives to deviate

Solution of a Game

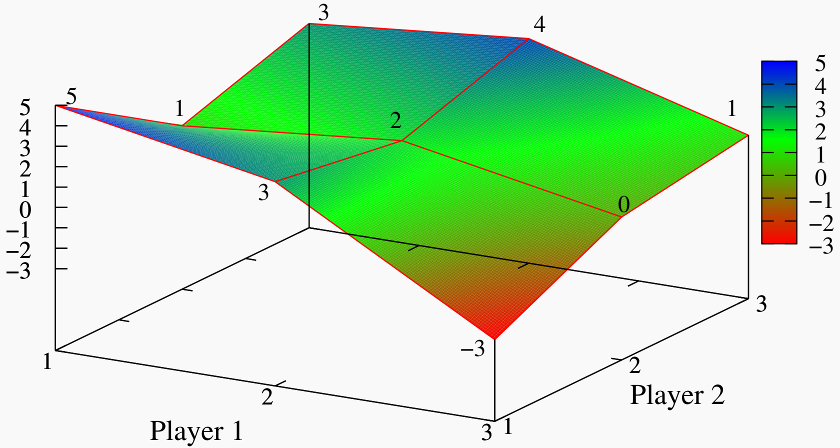
What is the solution of the game $\begin{array}{c|ccc|} \multicolumn{4}{c}{\text{Player 2}} \\ \cline{2-4} & 5 & 1 & 3 \\ \text{Player 1} & 3 & 2 & 4 \\ & -3 & 0 & 1 \\ \cline{2-4} \end{array}$?

Solution of a Game

What is the solution of the game $\begin{array}{c|ccc|} & \multicolumn{3}{c}{\text{Player 2}} \\ \cline{2-4} \text{Player 1} & 5 & 1 & 3 \\ & 3 & 2 & 4 \\ & -3 & 0 & 1 \\ \cline{2-4} \end{array}$?

Solution of a Game

Spatial Representation



Solution of a Game

What is the solution of the game $\begin{array}{c|ccc|} & \multicolumn{3}{c}{\text{Player 2}} \\ \cline{2-4} & 5 & 1 & 3 \\ \text{Player 1} & 3 & 2 & 4 \\ & -3 & 0 & 1 \\ \cline{2-4} \end{array}$?

Interpretation:

- Solution point is a saddle point
- Value of a game: $V = \underbrace{\min_j \max_i a_{ij}}_{V_+} = \underbrace{\max_i \min_j a_{ij}}_{V_-}$

Games with no solution?

Proposition:

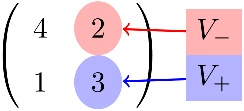
For any game, we can define:
$V_- = \max_i \min_j a_{ij}$ and $V_+ = \min_j \max_i a_{ij}$.

In general $V_- \leq V_+$

Proof.

$\forall i,\ \min_j \max_i a_{ij} \geq \min_j a_{ij}$ □

Example:



$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Interpretation of V_- and V_+

$$\begin{array}{|ccc|}\hline 4 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & 3 & 1 \\ \hline \end{array}$$

Interpretation 1: Security Strategy and Level

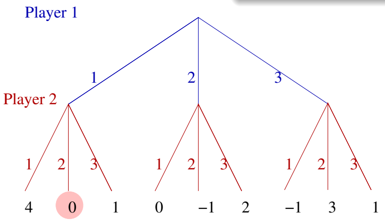
V_- is the utility that Player 1 can secure (“gain-floor”).
V_+ is the ”loss-ceiling” for Player 2.

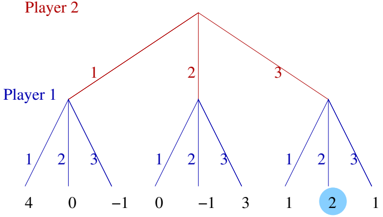
Interpretation of V_- and V_+

$$\begin{array}{|ccc|}\hline 4 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & 3 & 1 \\ \hline\end{array}$$

Interpretation 1: Security Strategy and Level

V_- is the utility that Player 1 can secure (“gain-floor”).
V_+ is the ”loss-ceiling” for Player 2.





Interpretation 2: Ordered Decision Making

Suppose that there is a predefined order in which players take decisions. (Then, whoever plays second has an advantage.)
V_- is the solution value when Player 1 plays first.
V_+ is the solution value when Player 2 plays first.

Games with more than one solution?

Proposition: Uniqueness of Solution

A zero-sum game admits a unique V_- and V_+. If it exists V is unique.
A zero-sum game admits at most one (strict) saddle point

Proof.

Let (i, j) and (k, l) be two saddle points. $\begin{pmatrix} a_{ij} & \cdots & a_{il} \\ & \vdots & \\ a_{kj} & \cdots & a_{kl} \end{pmatrix}$

By definition of a_{ij} : $a_{ij} \leq a_{il}$ and $a_{ij} \geq a_{kj}$. Similarly, by definition of a_{kl} : $a_{kl} \leq a_{kj}$ and $a_{kl} \geq a_{il}$.
Then, $a_{ij} \leq a_{il} \leq a_{kl} \leq a_{kj} \leq a_{ij}$ □

Extension to Mixed Strategies

Definition: Mixed Strategy.

A mixed strategy x is a probability distribution on the set of pure strategies: $\forall i, x_i \geq 0$, $\displaystyle\sum_i x_i = 1$

Optimal Strategies:

- Player 1 maximize its expected gain-floor with $x =$ argmax $\displaystyle\min_y xAy^t$.
- Player 2 minimizes its expected loss-ceiling with $y =$ argmin $\displaystyle\max_x xAy^t$.

Values of the game:

- $V_-^m = \displaystyle\max_x \min_y xAy^t = \max_x \min_j xA_{.j}$ and
- $V_+^m = \displaystyle\min_y \max_x xAy^t = \min_y \max_i A_{i.}y^t$.

The Minimax Theorem

Theorem 1: The Minimax Theorem.

In mixed strategies: $V_-^m = V_+^m \overset{\text{def}}{=} V^m$

Proof.

Lemma 1: Theorem of the Supporting Hyperplane.

Let B a closed and convex set of points in R^n and $x \notin B$ Then,
$\exists p_1, ...p_n, p_{n+1} : \displaystyle\sum_{i=1}^{n} x_i p_i = p_{n+1}$ and $\forall y \in B, p_{n+1} < \displaystyle\sum_{i=1}^{n} p_i y_i$.

Proof.

Consider z the point in B of minimum distance to x and consider
$\forall n, 1 \leq i \leq n, p_i = z_i - x_i$, $p_{n+1} = \displaystyle\sum_i z_i x_i - \sum_i x_i$ □

□

The Minimax Theorem

Theorem 1: **The Minimax Theorem**.

In mixed strategies: $V_-^m = V_+^m \overset{\text{def}}{=} V^m$

Proof.

Lemma 1: **Theorem of the Alternative for Matrices**.

Let $A = (a_{ij})_{m \times n}$ Either (i) $(0, ..., 0)$ is contained in the convex hull of $A_{.1}, ..., A_{.n}, e_1, ...e_m$. Or (ii) There exists $x_1, ..., x_m$ s.t. $\forall i, x_i > 0, \sum_{i=1}^{m} x_i = 1, \forall j \in 1, ..., n, \sum_{i=1}^{m} a_{ij} x_i$.

Lemma 2.

Lemma 3: Let A be a game and $k \in R$. Let B the game such that $\forall i, j, b_{ij} = a_{ij} + k$. Then $V_-^m(A) = V_-^m(B) + k$ and $V_+^m(A) = V_+^m(B) + k$.

The Minimax Theorem

Theorem 1: **The Minimax Theorem**.

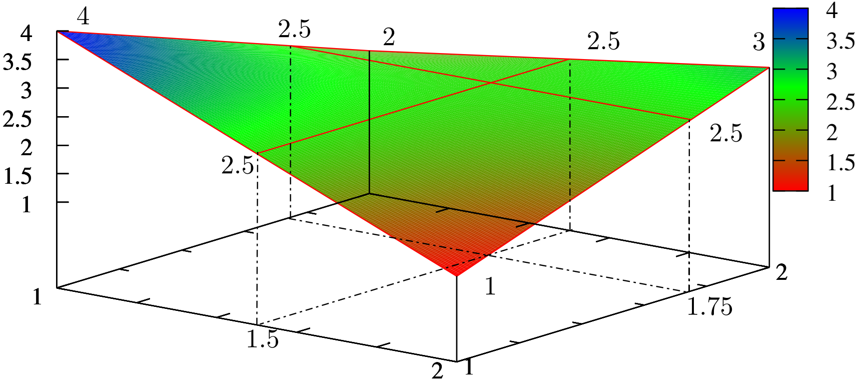
In mixed strategies: $V_-^m = V_+^m \overset{\text{def}}{=} V^m$

Proof.

From Lemma 2, we get that for any game, either (i) from lemma 2 and $V_+^m \leq 0$ or (ii) and $V_-^m > 0$. Hence, we cannot have $V_-^m \leq 0 < V_+^m$. With Lemma 3 this implies that $V_-^m = V_+^m$. □

The Minimax Theorem - Illustration

Example: $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$



A Note on Symmetric Games

Definition: **Symmetric Game**.

A game is symmetric if its matrix is skew-symmetric

Proposition:

The value of a symmetric game is 0 and any strategy optimal for player 1 is also optimal for player 2.

Proof.

Note that $xAx^t = -xA^tx^t = -(xAx^t)^t = -xAx^t = 0$. Hence $\forall x, \min_y xAy^t \leq 0$ and $\max_y yAx^t \geq 0$ so $V = 0$.

If x is an optimal strategy for 1 then $0 \leq xA = x(-A^t) = -xA^t$ and $Ax^t \leq 0$. □

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case
2. Two Inspiring Examples
3. Optimality
4. Bargaining Concepts
5. Measuring the Inefficiency of a Policy
6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

Game in Normal Form

Definition: (Finite or Matrix) Game.

- N players, finite number of actions
- Payoffs of players (depend of each other actions and) are real valued
- Stable points are called Nash Equilibria

Definition: Nash Equilibrium.

In a NE, no player has incentive to unilaterally modify his strategy.

strategy

payoff

s^* is a Nash equilibrium iff:

$$\forall p, \forall s_p, u_p(s_1^*, \dots, s_p^*, \dots s_n^*) \geq u_p(s_1^*, \dots, s_p, \dots, s_n^*)$$

In a compact form:

$$\forall p, \forall s_p, u_p(s_{-p}^*, s_p^*) \geq u_p(s_{-p}^*, s_p)$$

Nash Equilibrium: Examples

Why are these games be called “matrix” games?

Nash Equilibrium: Examples

Why are these games be called “matrix” games?

How many vector matrices (and of which size) need to be used to represent a game with N players where each player has M strategies?

Nash Equilibrium: Examples

Find the Nash equilibria of these games (with pure strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1, 1)$	$(3, 0)$
deny	$(0, 3)$	$(2, 2)$

Rock-Scisor-Paper

Nash Equilibrium: Examples

Find the Nash equilibria of these games (with pure strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

$\Rightarrow$ not efficient

Battle of the sexes

Paul / Claire	Opera	Foot
Opera	$(2,1)$	$(0,0)$
Foot	$(0,0)$	$(1,2)$

Rock-Scisor-Paper

$1/2$	P	R	S
P	$(0,0)$	$(1,-1)$	$(-1,1)$
R	$(-1,1)$	$(0,0)$	$(1,-1)$
S	$(1,-1)$	$(-1,1)$	$(0,0)$

Nash Equilibrium: Examples

Find the Nash equilibria of these games (with pure strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

⇒ not efficient

Battle of the sexes

Paul / Claire	Opera	Foot
Opera	$(2,1)$	$(0,0)$
Foot	$(0,0)$	$(1,2)$

⇒ not unique

Rock-Scisor-Paper

$1/2$	P	R	S
P	$(0,0)$	$(1,-1)$	$(-1,1)$
R	$(-1,1)$	$(0,0)$	$(1,-1)$
S	$(1,-1)$	$(-1,1)$	$(0,0)$

⇒ No equilibrium

Mixed Nash Equilibria

Definition: **Mixed Strategy Nash Equilibria**.

A mixed strategy for player i is a probability distribution over the set of pure strategies of player i.
An equilibrium in mixed strategies is a strategy profile σ^* of mixed strategies such that: $\forall p, \forall \sigma_i, u_p(\sigma^*_{-p}, \sigma^*_p) \geq u_p(\sigma^*_{-p}, \sigma_p)$.

Theorem 2.

Any finite n-person noncooperative game has at least one equilibrium n-tuple of mixed strategies.

Proof.

Kakutani fixed point theorem: Apply Kakutani to $f : \sigma \mapsto \otimes_{i \in \{1,N\}} B_i(\sigma_i)$ with $B_i(\sigma)$ the best response of player i. □

Mixed Nash Equilibria

Definition: Mixed Strategy Nash Equilibria.

A mixed strategy for player i is a probability distribution over the set of

Consequence:

- The players mixed strategies are independant randomizations.
- In a finite game, $u_p(\sigma) = \sum_a \left(\prod_{p'} \sigma_{p'}(a_{p'}) \right) u_i(a)$.
- Function u_i is multilinear
- In a finite game, σ^* is a Nash equilibrium iff $\forall a_i$ in the support of σ_i, a_i is a best response to σ^*_{-i}

Proof.

Kakutani fixed point theorem: Apply Kakutani to $f : \sigma \mapsto \otimes_{i \in \{1,N\}} B_i(\sigma_i)$ with $B_i(\sigma)$ the best response of player i. □

Mixed Nash Equilibria: Examples

Find the Nash equilibria of these games (with mixed strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

Mixed Nash Equilibria: Examples

Find the Nash equilibria of these games (with mixed strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

$\Rightarrow$ No strictly mixed equilibria

Battle of the sexes

Paul / Claire	Opera	Foot
Opera	$(2,1)$	$(0,0)$
Foot	$(0,0)$	$(1,2)$

Mixed Nash Equilibria: Examples

Find the Nash equilibria of these games (with mixed strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

⇒ No strictly mixed equilibria

Battle of the sexes

Paul / Claire	Opera	Foot
Opera	$(2,1)$	$(0,0)$
Foot	$(0,0)$	$(1,2)$

$\sigma_1 = (2/3, 1/3)$, $\sigma_2 = (1/3, 2/3)$

Rock-Scisor-Paper

Mixed Nash Equilibria: Examples

Find the Nash equilibria of these games (with mixed strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

⇒ No strictly mixed equilibria

Battle of the sexes

Paul / Claire	Opera	Foot
Opera	$(2,1)$	$(0,0)$
Foot	$(0,0)$	$(1,2)$

$\sigma_1 = (2/3, 1/3)$, $\sigma_2 = (1/3, 2/3)$

Rock-Scisor-Paper

$1/2$	P	R	S
P	$(0,0)$	$(1,-1)$	$(-1,1)$
R	$(-1,1)$	$(0,0)$	$(1,-1)$
S	$(1,-1)$	$(-1,1)$	$(0,0)$

Mixed Nash Equilibria: Examples

Find the Nash equilibria of these games (with mixed strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

⇒ No strictly mixed equilibria

Battle of the sexes

Paul / Claire	Opera	Foot
Opera	$(2,1)$	$(0,0)$
Foot	$(0,0)$	$(1,2)$

$\sigma_1 = (2/3, 1/3)$, $\sigma_2 = (1/3, 2/3)$

Rock-Scisor-Paper

$1/2$	P	R	S
P	$(0,0)$	$(1,-1)$	$(-1,1)$
R	$(-1,1)$	$(0,0)$	$(1,-1)$
S	$(1,-1)$	$(-1,1)$	$(0,0)$

$\sigma_1 = \sigma_2 = (1/3, 1/3, 1/3)$

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case

2. Two Inspiring Examples

3. Optimality

4. Bargaining Concepts

5. Measuring the Inefficiency of a Policy

6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

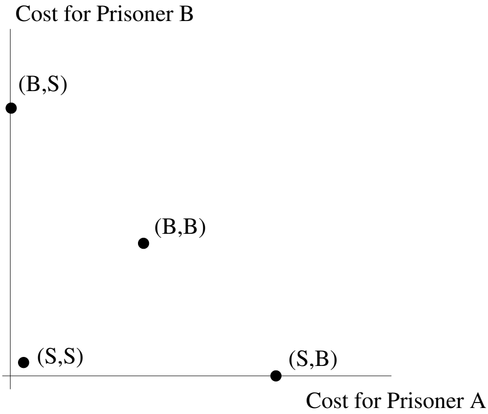
The Prisoner Dilemma

	Prisoner B stays Silent	Prisoner B Betrays
A stays Silent	Each serves 6 months	Prisoner A: 10 years Prisoner B: goes free
A Betrays	Prisoner A goes free Prisoner B: 10 years	Each serves 5 years

What is the best interest of each prisoner?

What is the output (Nash Equilibrium) of the game?

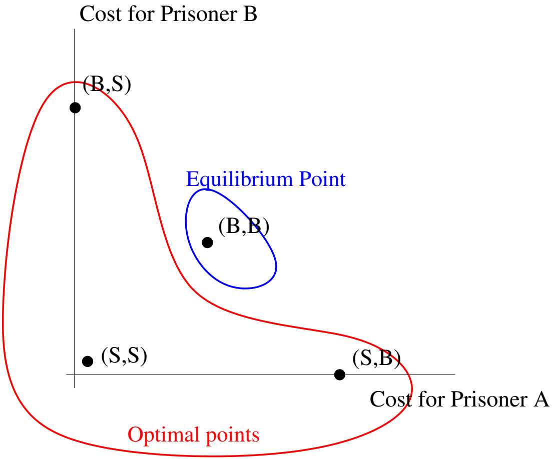
The Prisoner Dilemma - Cost Space



What are the optimal points?

What is the equilibrium?

The Prisoner Dilemma - Cost Space

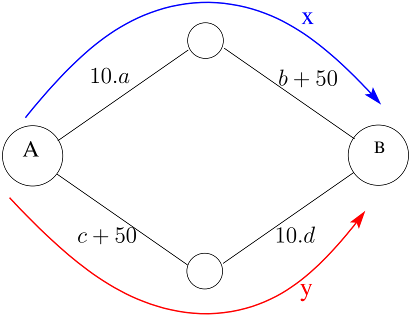


What are the optimal points?

What is the equilibrium?

The Braess Paradox

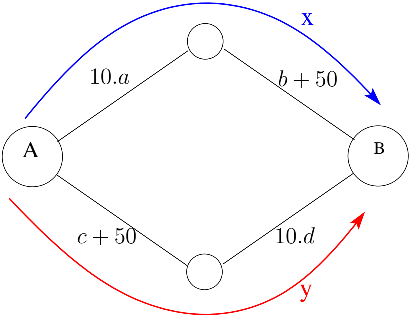
Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



- 2 possibles routes
- the needed time is a function of the number of cars on the road (congestion)

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?

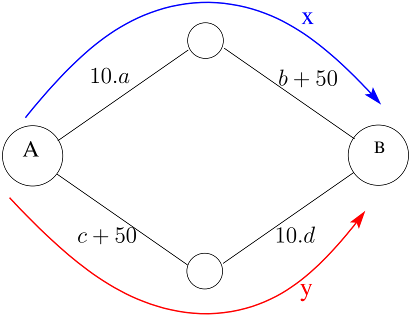


Which route will one take?

The one with minimum cost

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



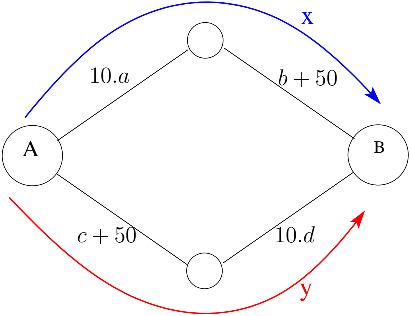
Which route will one take?

The one with minimum cost

Cost of route “north”:

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



Which route will one take?

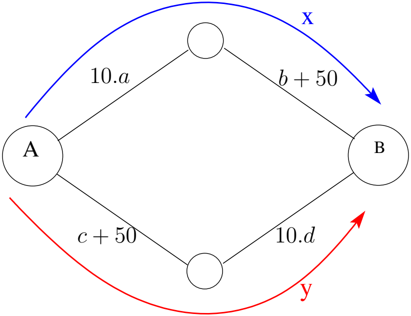
The one with minimum cost

Cost of route "north":

$$10 * x + (x + 50) = 11 * x + 50$$

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



Which route will one take?

The one with minimum cost

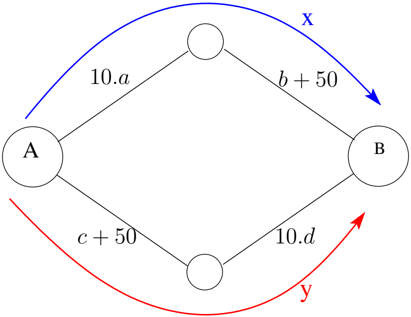
Cost of route “north”:

$10 * x + (x + 50) = 11 * x + 50$

Cost of route “south”:

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



Which route will one take?

The one with minimum cost

Cost of route “north”:

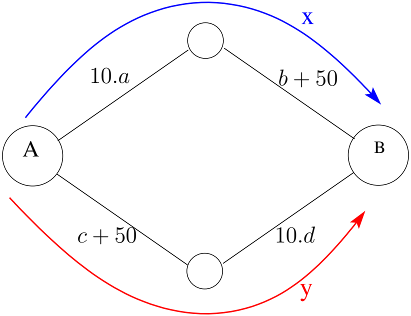
$10 * x + (x + 50) = 11 * x + 50$

Cost of route “south”:

$(y + 50) + 10 * y = 11 * y + 50$

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



Which route will one take?

The one with minimum cost

Cost of route "north":

$10 * x + (x + 50) = 11 * x + 50$

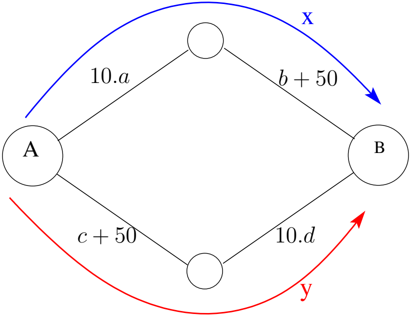
Cost of route "south":

$(y + 50) + 10 * y = 11 * y + 50$

Constraint: $x + y = 6$

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



Which route will one take?

The one with minimum cost

Cost of route "north":

$10 * x + (x + 50) = 11 * x + 50$

Cost of route "south":

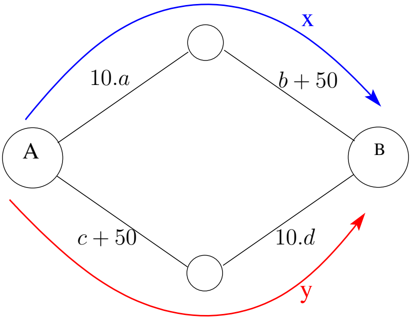
$(y + 50) + 10 * y = 11 * y + 50$

Constraint: $x + y = 6$

Conclusion? What if everyone makes the same reasoning?

The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



Which route will one take?

The one with minimum cost

Cost of route “north”:

$10 * x + (x + 50) = 11 * x + 50$

Cost of route “south”:

$(y + 50) + 10 * y = 11 * y + 50$

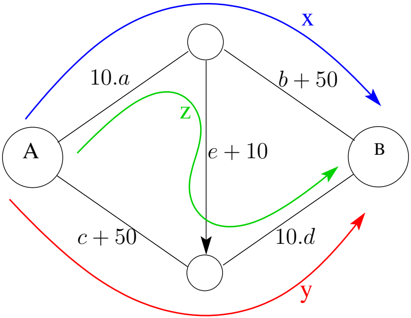
Constraint: $x + y = 6$

Conclusion? What if everyone makes the same reasoning?

We get $x = y = 3$ and everyone receives 83

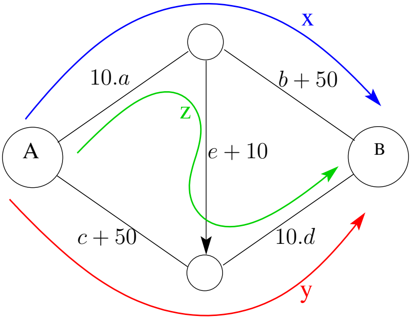
The Braess Paradox

A new road is opened! What happens?



The Braess Paradox

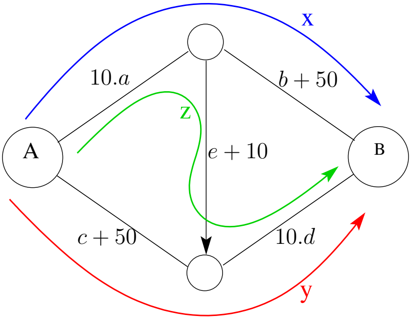
A new road is opened! What happens?



If noone takes it, it cost is

The Braess Paradox

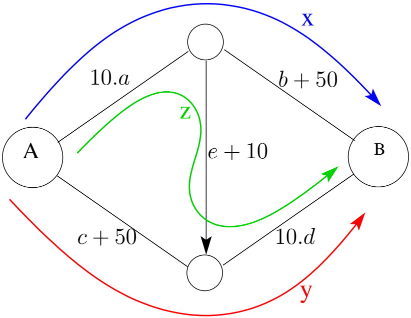
A new road is opened! What happens?



If noone takes it, it cost is 70! so rational users will take it... ☺

The Braess Paradox

A new road is opened! What happens?

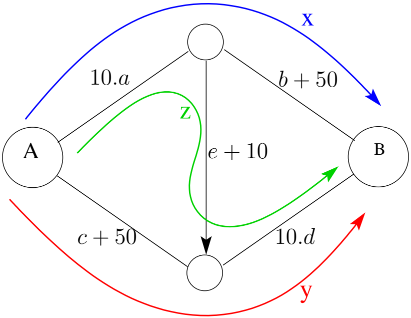


If noone takes it, it cost is 70! so rational users will take it... ☺

Cost of route“north”:

The Braess Paradox

A new road is opened! What happens?



If noone takes it, it cost is 70! so rational users will take it... ☺

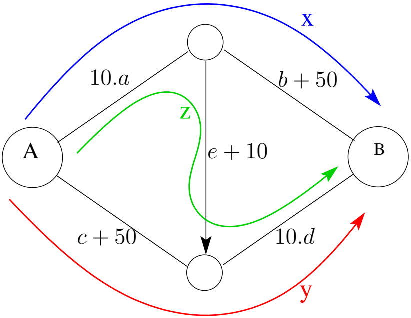
Cost of route“north”:

$10 * (x + z) + (x + 50) =$

$11 * x + 50 + 10 * z$

The Braess Paradox

A new road is opened! What happens?



If noone takes it, it cost is 70! so rational users will take it... ☺

Cost of route“north”:

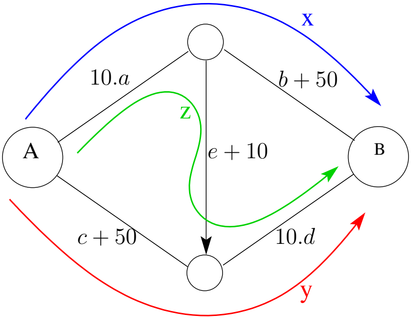
$10 * (x + z) + (x + 50) =$

$11 * x + 50 + 10 * z$

Cost of route “south”:

The Braess Paradox

A new road is opened! What happens?



If noone takes it, it cost is 70! so rational users will take it... ☺

Cost of route“north”:

$10 * (x + z) + (x + 50) =$

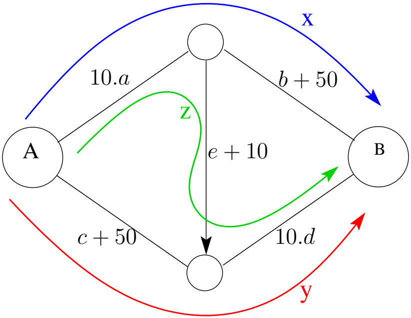
$11 * x + 50 + 10 * z$

Cost of route “south”:

$11 * y + 50 + 10 * z$

The Braess Paradox

A new road is opened! What happens?



If noone takes it, it cost is 70! so rational users will take it... ☺

Cost of route"north":

$10 * (x + z) + (x + 50) =$

$11 * x + 50 + 10 * z$

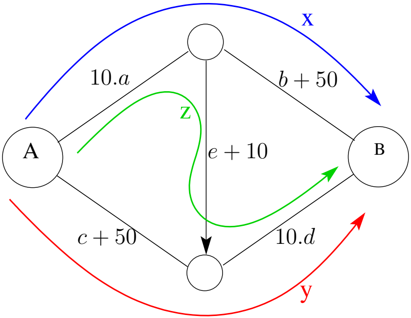
Cost of route "south":

$11 * y + 50 + 10 * z$

Cost of "new" route:

The Braess Paradox

A new road is opened! What happens?



If noone takes it, it cost is 70! so rational users will take it... ☺

Cost of route“north”:

$10*(x+z)+(x+50)=$

$11*x+50+10*z$

Cost of route “south”:

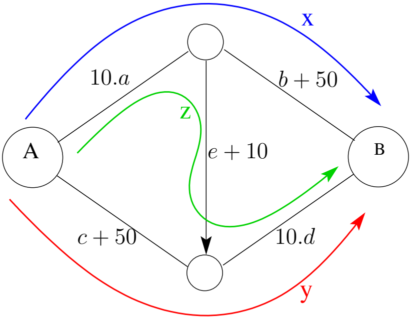
$11*y+50+10*z$

Cost of “new” route:

$10*x+10*y+21*z+10$

The Braess Paradox

A new road is opened! What happens?



If noone takes it, it cost is 70! so rational users will take it... ☺

Cost of route"north":

$10 * (x + z) + (x + 50) =$

$11 * x + 50 + 10 * z$

Cost of route "south":

$11 * y + 50 + 10 * z$

Cost of "new" route:

$10 * x + 10 * y + 21 * z + 10$

Conclusion?

We get $x = y = z = 2$ and everyone gets a cost of 92!

The Braess Paradox

In le New York Times, 25 Dec., 1990, Page 38, What if They Closed 42d Street and Nobody Noticed?, By GINA KOLATA:

ON Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem." But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.

Braess Paradox: Definition

Definition: **Braess-paradox**.

A Braess-paradoxes is a situation where exists two configurations S_1 and S_2 corresponding to utility sets $U(S_1)$ and $U(S_2)$ such that

$$U(S_1) \subset U(S_2) \text{ and } \forall k, \alpha_k(S_1) > \alpha_k(S_2)$$

with $\alpha(S)$ being the utility vector at equilibrium point for utility set S.

- In other words, in a Braess paradox, adding resource to the system decreases the utility of all players.
- Note that in systems where the equilibria are (Pareto) optimal, Braess paradoxes cannot occur.

Efficiency versus (Individual) Stability

Prisoner Dilemma / Braess paradox show:

- Inherent conflict between individual interest and global interest
- Inherent conflict between stability and optimality

Typical problem in economy: free-market economy versus regulated economy.

Efficiency versus (Individual) Stability

Free-Market:

Efficiency versus (Individual) Stability

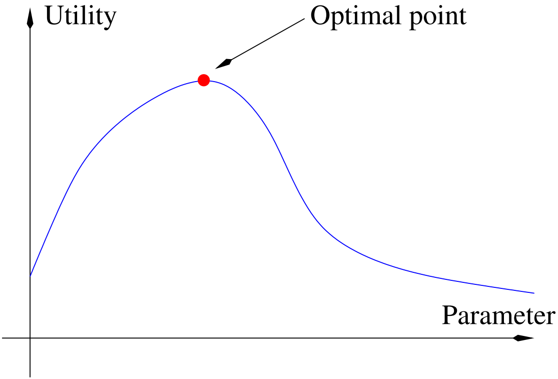
Regulated Market

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case
2. Two Inspiring Examples
3. Optimality
4. Bargaining Concepts
5. Measuring the Inefficiency of a Policy
6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

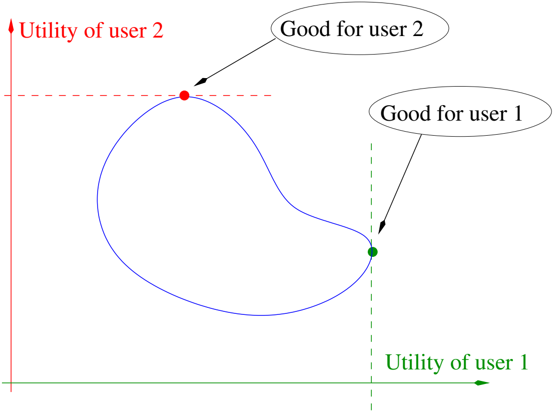
Defining Optimality in Multi-User Sytems

- Optimality for a single user



Defining Optimality in Multi-User Sytems

- Situation with multiple users



Analogy with: multi-criteria, hierarchical, zenith optimization.

Defining Optimality in Multi-User Sytems

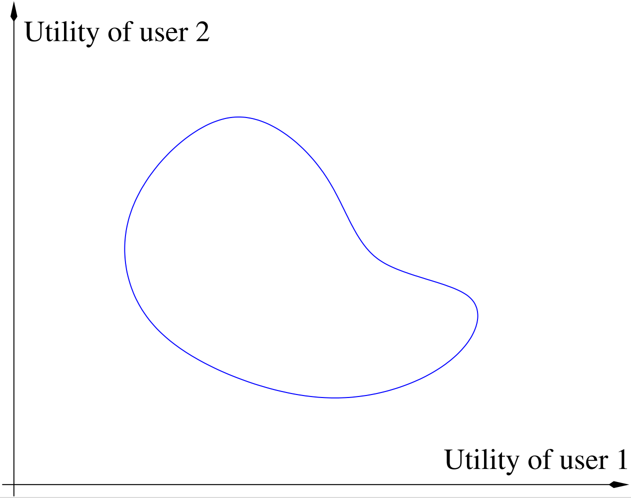
Definition: **Pareto Optimality**.

A point is said Pareto optimal if it cannot be strictly dominated by another.

Defining Optimality in Multi-User Sytems

Definition: **Pareto Optimality**.

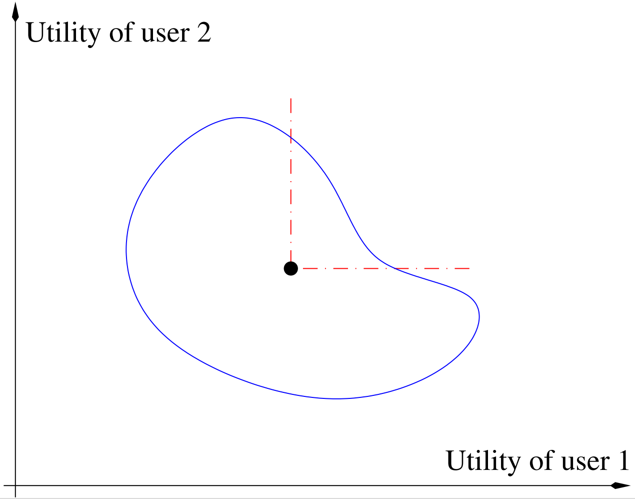
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

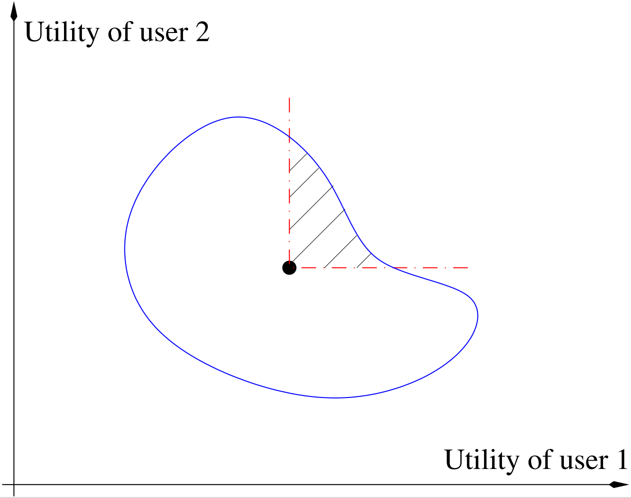
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

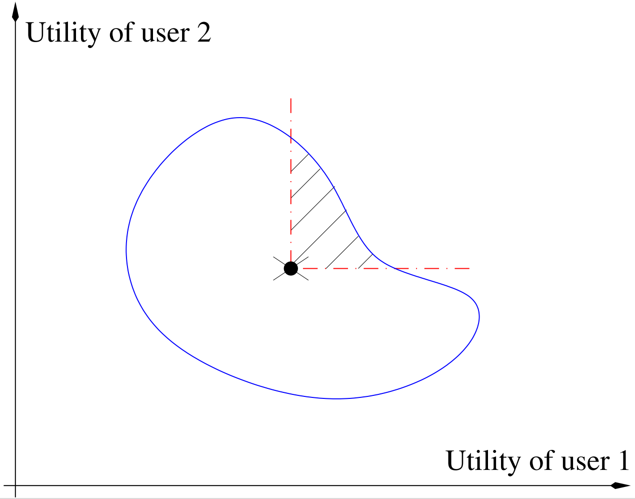
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: **Pareto Optimality**.

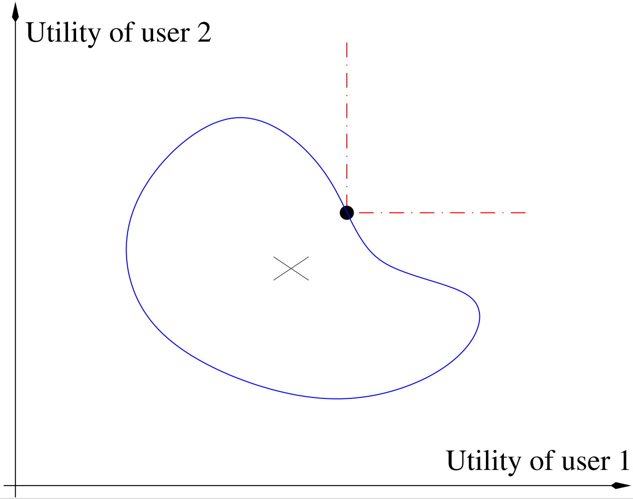
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

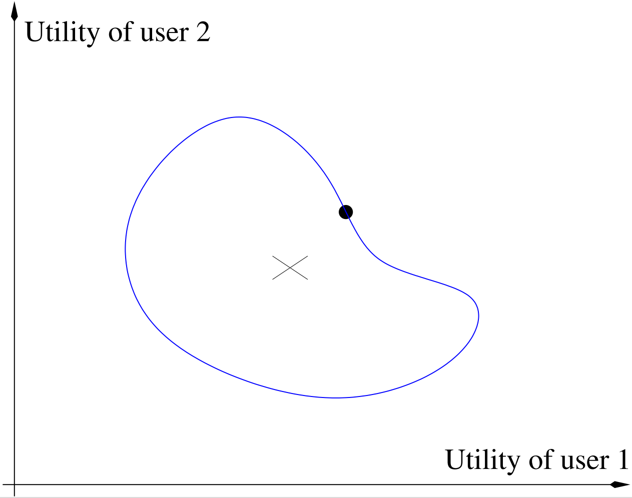
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

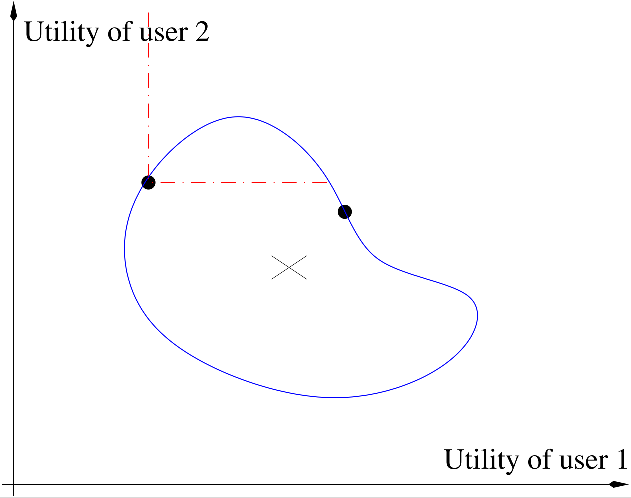
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

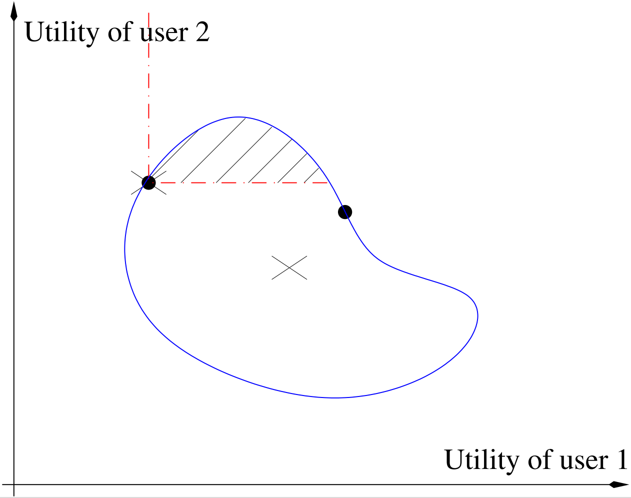
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

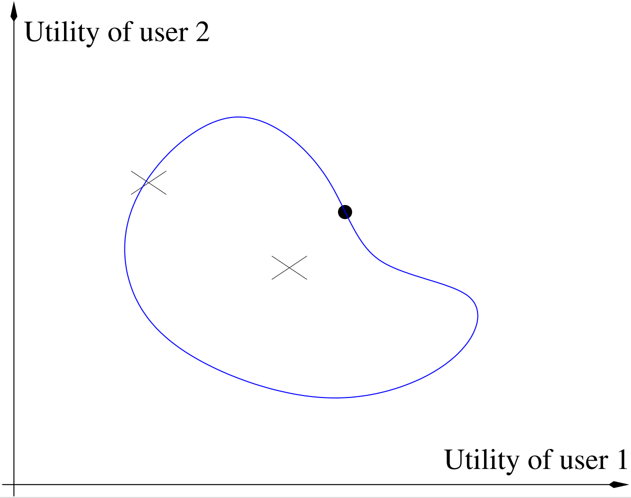
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

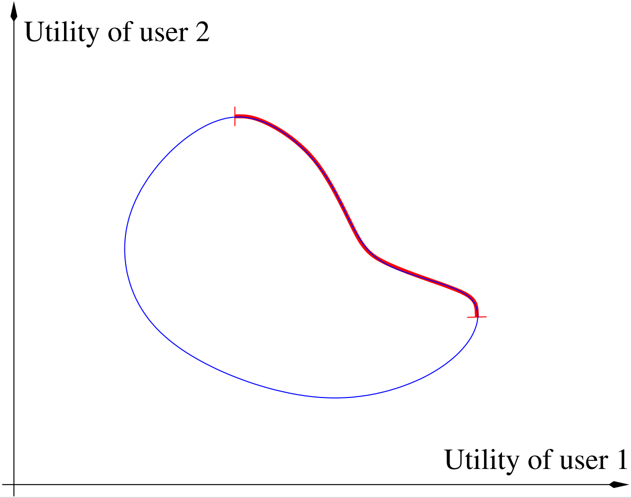
A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: Pareto Optimality.

A point is said Pareto optimal if it cannot be strictly dominated by another.



Defining Optimality in Multi-User Sytems

Definition: **Canonical order**.

We define the strict partial order $\ll$ on $\mathbb{R}_+^n$, namely the strict Pareto-superiority, by $u \ll v \Leftrightarrow \forall k : u_k \leq v_k$ and $\exists \ell, u_\ell < v_\ell$.

Definition: **Pareto optimality**.

A choice $u \in U$ is said to be Pareto optimal if it is maximal in U for the canonical partial order on $\mathbb{R}_+^n$.
A policy function α is said to be Pareto-optimal if $\forall U \in \mathcal{U}, \alpha(U)$ is Pareto-optimal.

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case
2. Two Inspiring Examples
3. Optimality
4. Bargaining Concepts
5. Measuring the Inefficiency of a Policy
6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

Bargaining Theory

- Aims at predicting the outcome of a bargain between 2 (or more) players
- The players are bargaining over a set of goods
- To each good is associated for each player a utility (for instance real valued)

Assumptions:

- Players have identical bargaining power
- Players have identical bargaining skills

Then, players will eventually agree on an point considered as “fair” for both of them.

The Nash Solution

Let S be a feasible set, closed, convex, (u^*, v^*) a point in this set, enforced if no agreement is reached.
A fair solution is a point $\phi(S, u^*, v^*)$ satisfying the set of axioms:

1. (Individual Rationality) $\phi(S, u^*, v^*) \geq (u^*, v^*)$ (componentwise)
2. (Feasibility) $\phi(S, u^*, v^*) \in S$
3. (Pareto-Optimality) $\forall (u, v) \in S, (u, v) \geq \phi(S, u^*, v^*) \rightarrow (u, v) = \phi(S, u^*, v^*)$
4. (Independence of Irrelevant Alternatives) $\phi(S, u^*, v^*) \in T \subset S \Rightarrow \phi(S, u^*, v^*) = \phi(T, u^*, v^*)$
5. (Independence of Linear Transformations) Let $F(u, v) = (\alpha_1 u + \beta_1, \alpha_2 v + \beta_2)$, $T = F(S)$, then $\phi(T, F(u^*, v^*)) = F(\phi(S, u^*, v^*))$
6. (Symmetry) If S is such that $(u, v) \in S \Leftrightarrow (v, u) \in S$ and $u^* = v^*$ then $\phi(S, u^*, v^*) \overset{\text{def}}{=} (a, b)$ is such that $a = b$

The Nash Solution

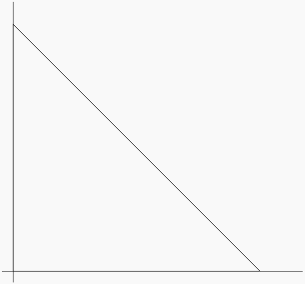
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case

□

The Nash Solution

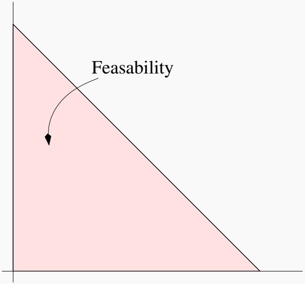
Proposition: **Nash Bargaining Solution**

There is a unique solution function ϕ satisfying all axioms:

$$\phi(S, u^*, v^*) = \max_{u,v} (u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case

□

The Nash Solution

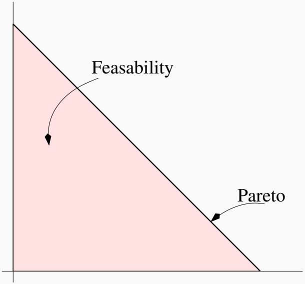
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case

□

The Nash Solution

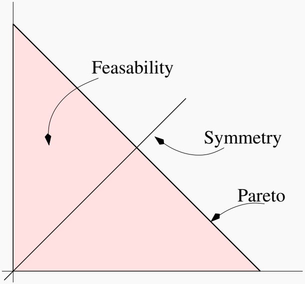
Proposition: **Nash Bargaining Solution**

There is a unique solution function ϕ satisfying all axioms:

$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case

□

The Nash Solution

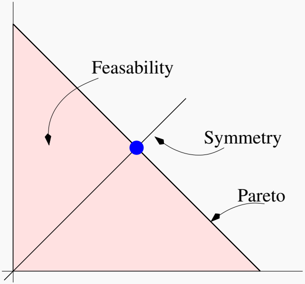
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

$$\phi(S, u^*, v^*) = \max_{u,v} (u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case

□

The Nash Solution

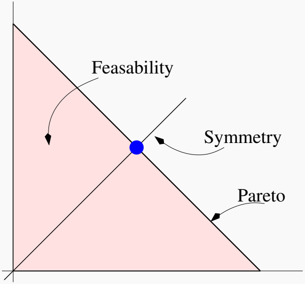
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

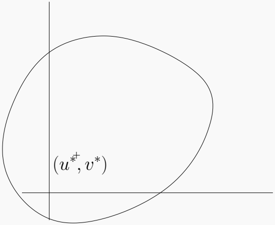
$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case



□

The Nash Solution

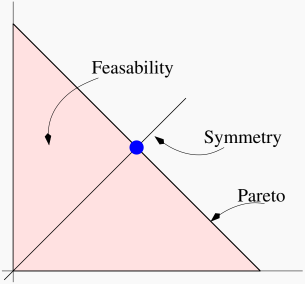
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

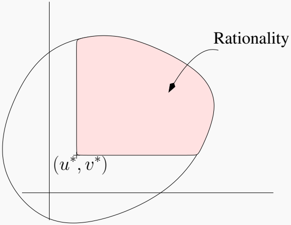
$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case



□

The Nash Solution

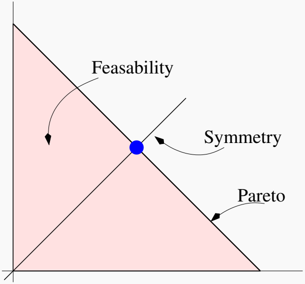
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

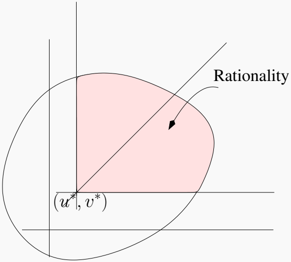
$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case



□

The Nash Solution

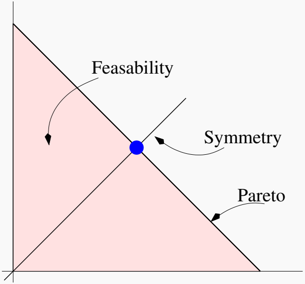
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

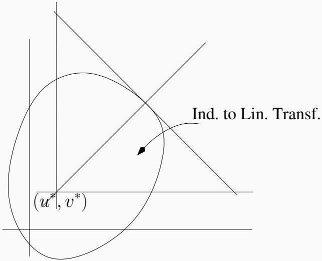
$$\phi(S, u^*, v^*) = \max_{u,v} (u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case



□

The Nash Solution

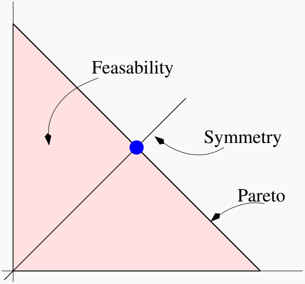
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

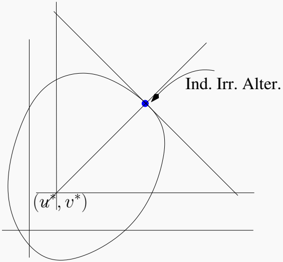
$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case



□

The Nash Solution

Example

Example (The Rich and the Poor Man)

- A rich man, a wealth of $\$1.000.000$
- A poor man, with a wealth of $\$100$
- A sum of $\$100$ to be shared between them. If they can't agree, none of them gets anything
- The utility to get some amount of money is the logarithm of the wealth growth
- How much should each one gets?

Let x be the sum going to the rich man.

$u(x) = \log\left(\frac{1000000+x}{1000000}\right) \sim \frac{x}{1000000}$ and $v(x) = \log\left(\frac{200-x}{100}\right)$.

The NBS is the solution of: $\max x \log\left(\frac{200-x}{100}\right)$, i.e.

$$x = \$54.5 \text{ and } \$45.5$$

the rich gets more! ☺

Axiomatic Definition VS Optimization Problem

1. Individual Rationality
2. Feasibility
3. Pareto-Optimality
5. Independence of Linear Transformations
6. Symmetry

Axiomatic Definition VS Optimization Problem

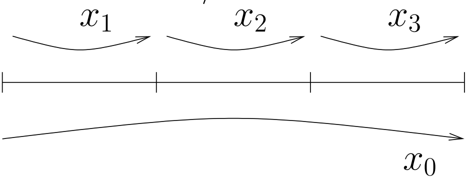
1. Individual Rationality
2. Feasibility
3. Pareto-Optimality
5. Independence of Linear Transformations
6. Symmetry

$+ \Bigg\{$

4. Independant to irrelevant alternatives Nash (NBS) / Proportional Fairness $\prod (u_i - u_i^d)$
4. Monotony Raiffa-Kalai-Smorodinsky / max-min Recursively $\max\{u_i | \forall j, u_i \leq u_j\}$
4. Inverse Monotony Thomson / global Optimum (Social welfare) $\max \sum u_i$

Example: The Flow Control Problem (1)

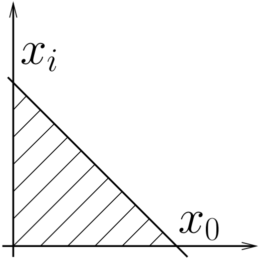
4 connections / 3 links.



$$\begin{cases} x_1 + x_0 \leq 1, \\ x_2 + x_0 \leq 1, \\ x_3 + x_0 \leq 1. \end{cases}$$

⇒ 4 variables and 3 (in)equalities.

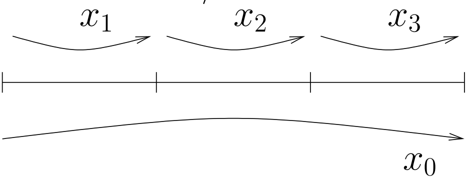
How to choose x_0 among the Pareto optimal points?



(Nota: in this case the utility set is the same as the strategy set)

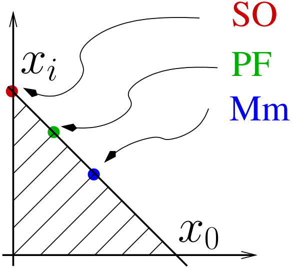
Example: The Flow Control Problem (1)

4 connections / 3 links.



$$\begin{cases} x_1 + x_0 \leq 1, \\ x_2 + x_0 \leq 1, \\ x_3 + x_0 \leq 1. \end{cases}$$

$\Rightarrow$ 4 variables and 3 (in)equalities.



How to choose x_0 among the Pareto optimal points?

$$\begin{cases} x_0 = 0.5, \\ x_1 = x_2 = x_3 = 0.5 \end{cases}$$ Max-Min fairness

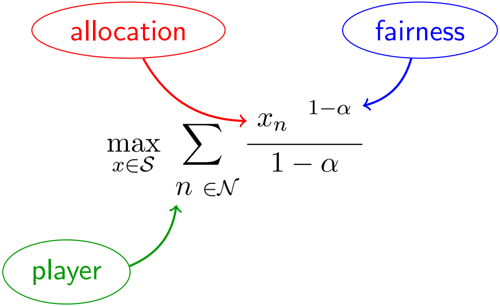
$$\begin{cases} x_0 = 0, \\ x_1 = x_2 = x_3 = 1 \end{cases}$$ Social Optimum

$$\begin{cases} x_0 = 0.25, \\ x_1 = x_2 = x_3 = 0.75 \end{cases}$$ Proportionnal Fairness

(Nota: in this case the utility set is the same as the strategy set)

Example: The Flow Control Problem (2)

Fairness family proposed by Mo and Walrand:

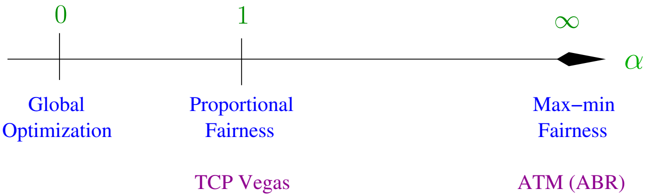


allocation

fairness

$$\max_{x\in\mathcal{S}} \sum_{n \in\mathcal{N}} \frac{x_n^{\;1-\alpha}}{1-\alpha}$$

player



0

1

∞

α

Global Optimization

Proportional Fairness

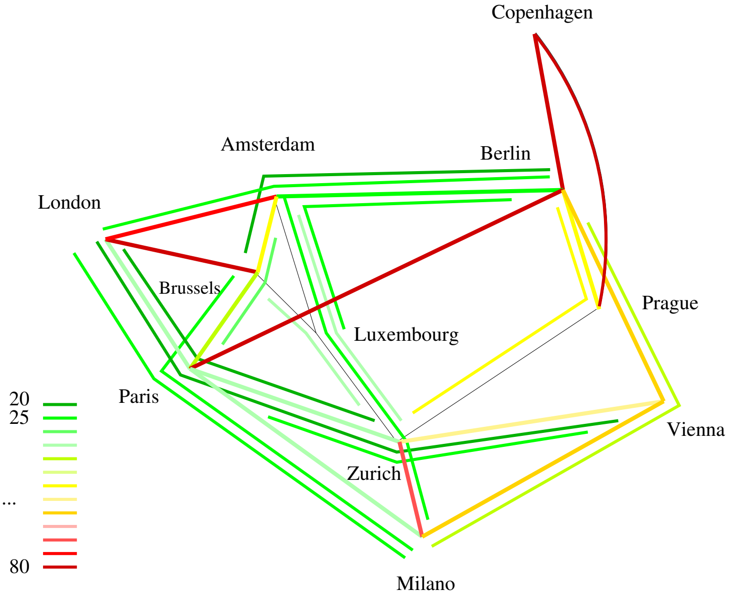
Max–min Fairness

TCP Vegas

ATM (ABR)

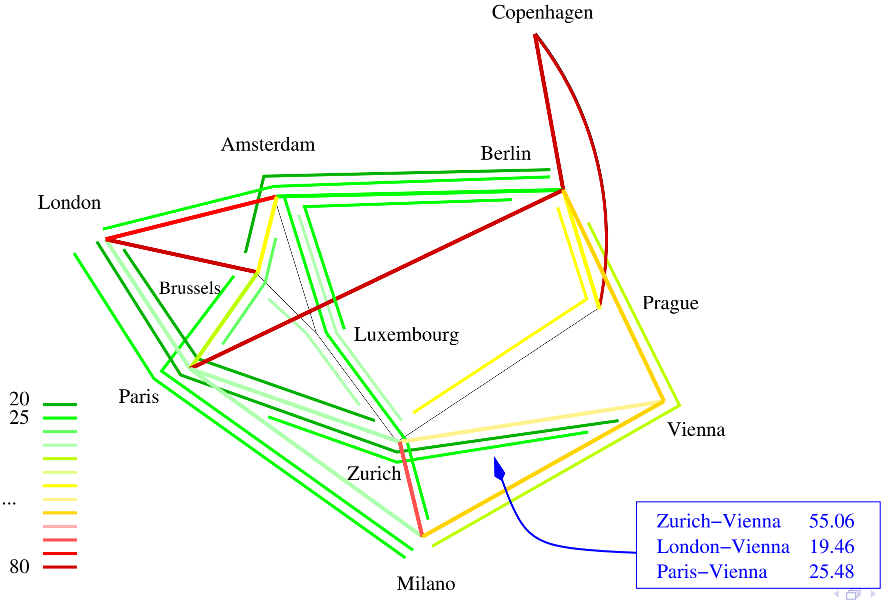
Example: The Flow Control Problem (3)

The COST network (Prop. Fairness.)



Example: The Flow Control Problem (3)

The COST network (Prop. Fairness.)



Zurich–Vienna	55.06
London–Vienna	19.46
Paris–Vienna	25.48

Example: The Flow Control Problem (4)

Problem is to maximize:

$$\max_x \underbrace{\boxed{\sum_n f_n(x_n)}}_{\boxed{\text{fairness aggregation function}}} \qquad \text{s.t.} \underbrace{\boxed{\forall \ell, (Ax)_\ell \leq C_l \text{ and } x \geq 0}}_{\boxed{\text{system constraints}}}$$

How to (efficiently and in a distributed manner) solve this?

Example: The Flow Control Problem (4)

Problem is to maximize:

$$\max_x \underbrace{\boxed{\sum_n f_n(x_n)}}_{\boxed{\text{fairness aggregation function}}} \qquad \text{s.t.} \underbrace{\boxed{\forall \ell, (Ax)_\ell \leq C_l \text{ and } x \geq 0}}_{\boxed{\text{system constraints}}}$$

How to (efficiently and in a distributed manner) solve this?

Answers in lecture 2 and 3 ☺

Time-Restricted Bargaining (Binmore & Rubinstein)

Context

- Feasible S, closed, convex.
- The bargaining process consists of rounds (1 round $= 1$ offer $+ 1$ counter-offer)
- If the two players can never agree, they receive a payoff $(0,0)$
- Each player has a discount factor (impatience) $\delta_i = e^{-a_i T}$

Solution

- A strategy for a player is a pair (a^*, b^{**}): he offers a^* to the other player and would accept any offer greater than b^{**}.
- A stationary equilibrium is a pair of strategies $((v^*, u^{**}), (u^*, v^{**}))$ such that both (u^*, v^*) and (u^{**}, v^{**}) are Pareto-optimal and $u^{**} = \delta_1 u^*$ and $v^* = \delta_2 v^{**}$.
- The stationary equilibrium exists and is unique.
- In the limit case $T \to 0$, then $(u^*, v^*) = (u^{**}, v^{**}) = \max\limits_{(u,v)\in S} uv^{a_1/a_2}$.

Note: Properties of the Fairness Family (1)

Theorem 3: Fairness and Optimality.

Let α be an f-optimizing policy. If f is strictly monotone then α is Pareto-optimal.

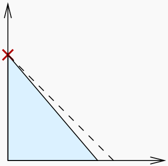
$\Rightarrow$ All Walrand & Mo family policies are Pareto

Let α be an f-optimizing policy. If α is Pareto-optimal then f is monotone.

Theorem 4: Continuity.

There exists continuous and non-continuous convex Pareto-optimal policy functions.

Example



Sum-optimizing is discontinuous, but the geometric mean-optimizing policy is continuous.

Note: Properties of the Fairness Family (1)

Theorem 3: **Fairness and Optimality**.

Let α be an f-optimizing policy. If f is strictly monotone then α is Pareto-optimal.

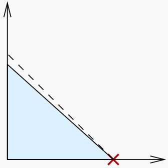
$\Rightarrow$ All Walrand & Mo family policies are Pareto

Let α be an f-optimizing policy. If α is Pareto-optimal then f is monotone.

Theorem 4: **Continuity**.

There exists continuous and non-continuous convex Pareto-optimal policy functions.

Example



Sum-optimizing is discontinuous, but the geometric mean-optimizing policy is continuous.

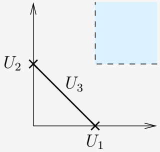
Note: Properties of the Fairness Family (2)

Theorem 5: **To have one's cake and eat it, too**.

A policy optimizing an index f is always non-monotone for a distinct index g.

⇒ allocations that are efficient (optimizing the arithmetic mean) cannot (in general) also be fair (optimizing the geometric mean).

Theorem 6: **Monotonicity**.



Even in convex sets, policy functions cannot be monotone.

⇒ even in Braess-free systems, an increase in the resource can be detrimental to some users.

[See. A. Legrand and C. Touati *"How to measure efficiency?"* Game-Comm'07, 2007 for details and proofs]

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case

2. Two Inspiring Examples

3. Optimality

4. Bargaining Concepts

5. Measuring the Inefficiency of a Policy

6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

Why is is important to develop inefficiency measures?

Why is is important to develop inefficiency measures?

Suppose that you are a network operator.
The different users compete to access the different system resources
Should you intervene?

- NO if the Nash Equilibria exhibit good performance
- YES otherwise

Why is is important to develop inefficiency measures?

Suppose that you are a network operator.
The different users compete to access the different system resources
Should you intervene?

- NO if the Nash Equilibria exhibit good performance
- YES otherwise

The question of “how” to intervene is the object of lecture 4 ☺

Why is is important to develop inefficiency measures?

Example: traffic lights would be useful ☺

Price of anarchy

For a given index f, let us consider $\alpha^{(f)}$ an f-optimizing policy function. We define the inefficiency $I_f(\beta, U)$ of the allocation $\beta(U)$ for f as

$$I_f(\beta, U) = \frac{f(\alpha^{(f)}_{(,U})}{f(\beta(U))} = \max_{u \in U} \frac{f(u)}{f(\beta(U))} \geq 1$$

Price of anarchy

For a given index f, let us consider $\alpha^{(f)}$ an f-optimizing policy function. We define the inefficiency $I_f(\beta, U)$ of the allocation $\beta(U)$ for f as

$$I_f(\beta, U) = \frac{f(\alpha^{(f)}_{(,U}))}{f(\beta(U))} = \max_{u \in U} \frac{f(u)}{f(\beta(U))} \geq 1$$

Papadimitriou focuses on the arithmetic mean Σ defined by

$$\Sigma(u_1, \ldots, u_k) = \sum_{k=1}^{K} u_k$$

The price of anarchy ϕ_Σ is thus defined as the largest inefficiency:

$$\phi_\Sigma(\beta) = \sup_{U \in \mathcal{U}} I_f(\beta, U) = \sup_{U \in \mathcal{U}} \frac{\sum_k \alpha^{(\Sigma)}_{(,U})_k}{\sum_k \beta(U)_k}$$

Price of anarchy

For a given index f, let us consider $\alpha^{(f)}$ an f-optimizing policy function. We define the inefficiency $I_f(\beta, U)$ of the allocation $\beta(U)$ for f as

$$I_f(\beta, U) = \frac{f(\alpha^{(f)}_{(,U})}{f(\beta(U))} = \max_{u \in U} \frac{f(u)}{f(\beta(U))} \geq 1$$

Papadimitriou focuses on the arithmetic mean Σ defined by

$$\Sigma(u_1, \dots, u_k) = \sum_{k=1}^{K} u_k$$

The price of anarchy ϕ_Σ is thus defined as the largest inefficiency:

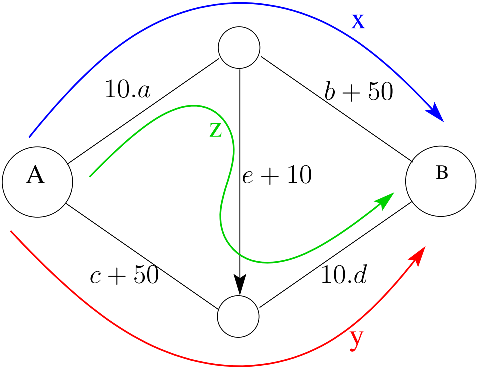
$$\phi_\Sigma(\beta) = \sup_{U \in \mathcal{U}} I_f(\beta, U) = \sup_{U \in \mathcal{U}} \frac{\sum_k \alpha^{(\Sigma)}_{(,U})_k}{\sum_k \beta(U)_k}$$

In other words, $\phi_\Sigma(\beta)$ is the approximation ratio of β for the objective function Σ.

Price of Anarchy: Example of Application

A routing problem is a triplet:

- A graph $G = (N, A)$ (the network)
- A set of flows d_k, $k \in K$ and $K \subset N \times N$ (user demands)
- latency funtions ℓ_a for each link



Theorem 7.

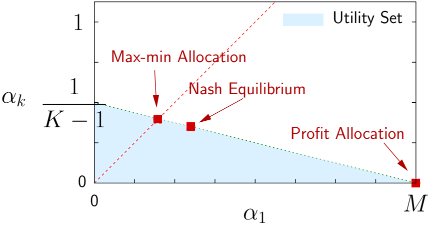
In networks with affine costs [Roughgarden & Tardos, 2002],

$$C^{WE} \leq \frac{4}{3} C^{SO}.$$

$\Rightarrow$ In affine routing, selfishness leads to a near optimal point.

Price of anarchy: does it really reflects inefficiencies?

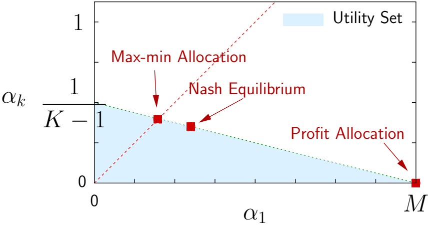
Consider the utility set $S_{M,N} = \{u \in \mathbb{R}_+^N | \frac{u_1}{M} + \sum_{k=1}^{N} u_k \leq 1\}$. As the roles of the u_k, $k \geq 2$ are symmetric, we can freely assume that $u_2 = \cdots = u_N$ for index-optimizing policies ([Legrand et al, Infocom'07]).



Utility set and allocations for $S_{M,N}$ $(N = 3, M = 2)$, with $u_2 = \cdots = u_N$.

Price of anarchy: does it really reflects inefficiencies?

Consider the utility set $S_{M,N} = \{u \in \mathbb{R}_+^N | \frac{u_1}{M} + \sum_{k=1}^{N} u_k \leq 1\}$. As the roles of the u_k, $k \geq 2$ are symmetric, we can freely assume that $u_2 = \cdots = u_N$ for index-optimizing policies ([Legrand et al, Infocom'07]).



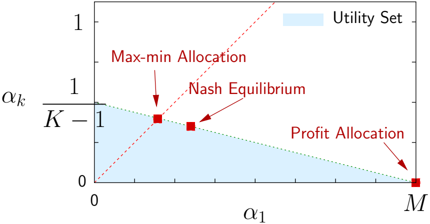
Utility set and allocations for $S_{M,N}$ $(N = 3, M = 2)$, with $u_2 = \cdots = u_N$.

$$I_\Sigma(\alpha^{\text{NBS}}, S_{M,N}) \xrightarrow[M\to\infty]{} N$$

$$I_\Sigma(\alpha^{\text{Max-Min}}, S_{M,N}) \sim_{M\to\infty} M$$

Price of anarchy: does it really reflects inefficiencies?

Consider the utility set $S_{M,N} = \{u \in \mathbb{R}_+^N | \frac{u_1}{M} + \sum_{k=1}^{N} u_k \leq 1\}$. As the roles of the u_k, $k \geq 2$ are symmetric, we can freely assume that $u_2 = \cdots = u_N$ for index-optimizing policies ([Legrand et al, Infocom'07]).



Utility set and allocations for $S_{M,N}$ $(N = 3, M = 2)$, with $u_2 = \cdots = u_N$.

$$I_\Sigma(\alpha^{\text{NBS}}, S_{M,N}) \xrightarrow[M\to\infty]{} N$$

$$I_\Sigma(\alpha^{\text{Max-Min}}, S_{M,N}) \sim_{M\to\infty} M$$

These are due to the fact that a policy optimizing an index f is always non-monotone for a distinct index g.

$\rightsquigarrow$ Pareto inefficiency should be measured as the distance to the Pareto border and not to a specific point.

Selfish Degradation Factor: A Definition

- The distance from $\beta(U)$ to the closure of the Pareto set $\overline{\mathcal{P}}(U)$ in the log-space is equal to:

$$d_\infty(\log(\beta(U), \log(\overline{\mathcal{P}}(U))) = \min_{u \in \overline{\mathcal{P}}(u)} \max_k |\log(\beta(U)_k) - \log(u_k)|$$

Therefore, we can define

$$\begin{aligned} \widetilde{I}_\infty(\beta, U) &= \exp(d_\infty(\log(\beta(U), \log(\overline{\mathcal{P}}(U))) \\ &= \min_{u \in \overline{\mathcal{P}}(u)} \max_k \max\left(\frac{\beta(U)_k}{u_k}, \frac{u_k}{\beta(U)_k}\right) \end{aligned}$$

[See A. Legrand, C. Touati, *How to measure efficiency?* Gamecom 2007, for a more detailed description and a topological discussion.]

Outline

1. "Simple" Games and their solutions: One Round, Simultaneous plays, Perfect Information
   - Zero-Sum Games
   - General Case
2. Two Inspiring Examples
3. Optimality
4. Bargaining Concepts
5. Measuring the Inefficiency of a Policy
6. Application: Multiple Bag-of-Task Applications in Distributed Platforms

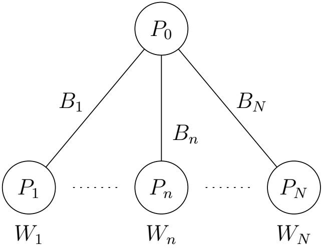
Application: Multiple Bag-of-Task Applications in Distributed Platforms

A number of concepts have been introduced to measure both efficiency and optimality of resource allocation. Yet, distributed platforms result from the collaboration of many users:

- Multiple applications execute concurrently on heterogeneous platforms and compete for CPU and network resources.
- Sharing resources amongst users should somehow be fair. In a grid context, this sharing is generally done in the "low" layers (network, OS).
- We analyze the behavior of K non-cooperative schedulers that use the optimal strategy to maximize their own utility while fair sharing is ensured at a system level ignoring applications characteristics.

Reference: A. Legrand, C. Touati, *"Non-cooperative scheduling of multiple bag-of-task applications"*, Infocom 2007.

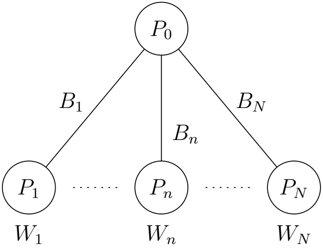
Master-Worker Platform



- N processors with processing capabilities W_n (in Mflop.s^{-1})
- using links with capacity B_n (in Mb.s^{-1})

Hypotheses :

Master-Worker Platform



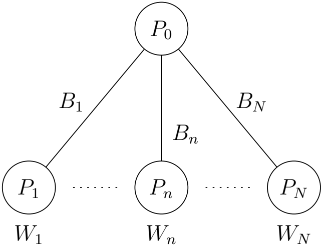
- N processors with processing capabilities W_n (in Mflop.s^{-1})
- using links with capacity B_n (in Mb.s^{-1})

Hypotheses :

- Multi-port

Communications to P_i do not interfere with communications to P_j.

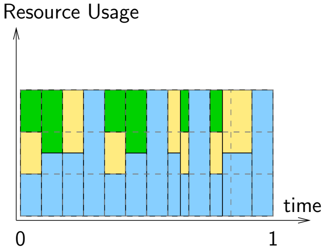
Master-Worker Platform



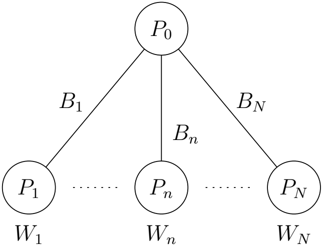
- N processors with processing capabilities W_n (in Mflop.s^{-1})
- using links with capacity B_n (in Mb.s^{-1})

Hypotheses :

- Multi-port
- No admission policy but an ideal local fair sharing of resources among the various requests



Master-Worker Platform



- N processors with processing capabilities W_n (in Mflop.s^{-1})
- using links with capacity B_n (in Mb.s^{-1})

Hypotheses :

Multi

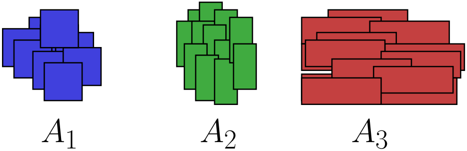
Definition.

We denote by physical-system a triplet (N, B, W) where N is the number of machines, and B and W the vectors of size N containing the link capacities and the computational powers of the machines.

requests

Applications

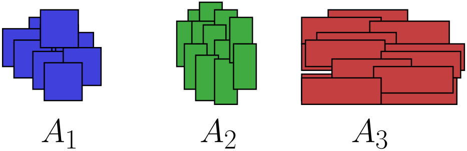
- Multiple applications $(A_1, \ldots, A_K)$:



  - each consisting in a large number of same-size independent tasks
  - Different communication and computation demands for different applications. For each task of A_k:
    - processing cost w_k (MFlops)
    - communication cost b_k (MBytes)

Applications

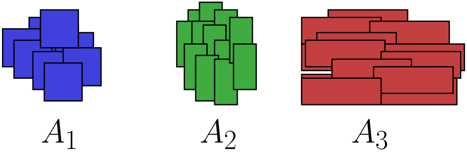
- Multiple applications $(A_1, \ldots, A_K)$:



  - each consisting in a large number of same-size independent tasks
  - Different communication and computation demands for different applications. For each task of A_k:
    - processing cost w_k (MFlops)
    - communication cost b_k (MBytes)
- Master holds all tasks initially, communication for input data only (no result message).

Applications

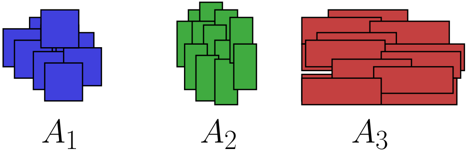
- Multiple applications $(A_1, \ldots, A_K)$:



  - each consisting in a large number of same-size independent tasks
  - Different communication and computation demands for different applications. For each task of A_k:
    - processing cost w_k (MFlops)
    - communication cost b_k (MBytes)
- Master holds all tasks initially, communication for input data only (no result message).
- Such applications are typical desktop grid applications (SETI@home, Einstein@Home, ...)

Applications

- Multiple applications $(A_1, \ldots, A_K)$:



  - each consisting in a large number of same-size independent tasks
  - Different communication and computation demands for different applications. For each task of A_k:
    - processing cost w_k (MFlops)
    - communication cost b_k (MBytes)

Definition.

We define an application-system as a triplet (K, b, w) where K is the number of applications, and b and w the vectors of size K representing the size and the amount of computation associated to the different applications.

Steady-State scheduling

In the following our K applications run on our N workers and compete for network and CPU access:

Definition.

A system S is a sextuplet (K, b, w, N, B, W), with K,b,w,N,B,W defined as for a user-system and a physical-system.

- Task regularity $\rightsquigarrow$ steady-state scheduling.
- Maximize throughput (average number of tasks processed per unit of time)

  $$\alpha_k = \lim_{t \to \infty} \frac{done_k(t)}{t}.$$

  Similarly: $\alpha_{n,k}$ is the average number of tasks of type k performed per time-unit on the processor P_n.

  $$\alpha_k = \sum_n \alpha_{n,k}.$$
- α_k is the utility of application k.

Constraints

The scheduler of each application thus aims at maximizing its own throughput, i.e. α_k.

However, as applications use the same set of resources, we have the following general constraints:

Computation $$\forall n \in [\![0, N]\!] : \sum_{k=1}^{K} \alpha_{n,k} \cdot w_k \leq W_n$$

Communication $$\forall n \in [\![1, N]\!] : \sum_{k=1}^{K} \alpha_{n,k} \cdot b_k \leq B_n$$

Applications should decide when to send data from the master to a worker and when to use a worker for computation.

Optimal strategy for a single application

Single application

This problem reduces to maximizing $\sum_{n=1}^{N} \alpha_{n,1}$ while:

$$\begin{cases} \forall n \in [\![A, N]\!] : \alpha_{n,1} \cdot w_1 \leq W_n \\ \forall n \in [\![1, N]\!] : \alpha_{n,1} \cdot b_1 \leq B_n \\ \forall n, \quad \alpha_{n,1} \geq 0. \end{cases}$$

The optimal solution to this linear program is obtained by setting

$$\forall n,\ \alpha_{n,1} = \min\left(\frac{W_n}{w_1}, \frac{B_n}{b_1}\right)$$

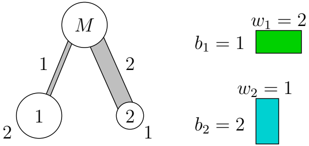
In other words

The master process should saturate each worker by sending it as many tasks as possible.

A simple acknowledgment mechanism enables the master process to ensure that it is not over-flooding the workers, while always converging to the optimal throughput.

A simple example

Two computers 1 and 2: $B_1 = 1$, $W_1 = 2$, $B_2 = 2$, $W_2 = 1$.
Two applications: $b_1 = 1$, $w_1 = 2$, $b_2 = 2$ and $w_2 = 1$.

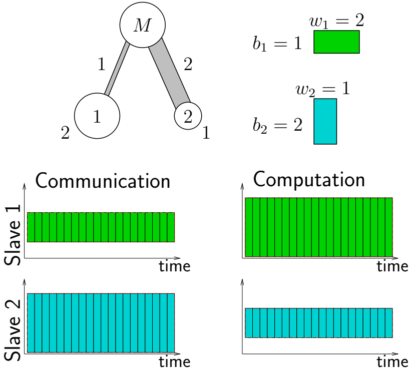


A simple example

Two computers 1 and 2: $B_1 = 1$, $W_1 = 2$, $B_2 = 2$, $W_2 = 1$. Two applications: $b_1 = 1$, $w_1 = 2$, $b_2 = 2$ and $w_2 = 1$.

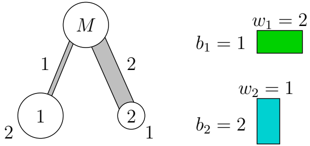
Cooperative Approach:

Application 1 is processed exclusively on computer 1 and application 2 on computer 2. The respective throughput is $\alpha_1^{(\text{coop})} = \alpha_2^{(\text{coop})} = 1$.



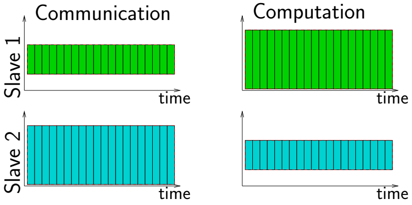
A simple example

Two computers 1 and 2: $B_1 = 1$, $W_1 = 2$, $B_2 = 2$, $W_2 = 1$. Two applications: $b_1 = 1$, $w_1 = 2$, $b_2 = 2$ and $w_2 = 1$.



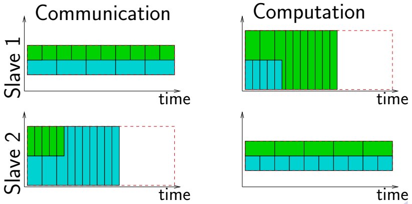
Cooperative Approach:

Application 1 is processed exclusively on computer 1 and application 2 on computer 2. The respective throughput is $\alpha_1^{(\text{coop})} = \alpha_2^{(\text{coop})} = 1$.



Non-Cooperative Approach:

$$\alpha_1^{(nc)} = \alpha_2^{(nc)} = \frac{3}{4}$$



A simple example

Nota: The "Divide and Conquer" philosophy does not apply to the definition of Pareto optimality

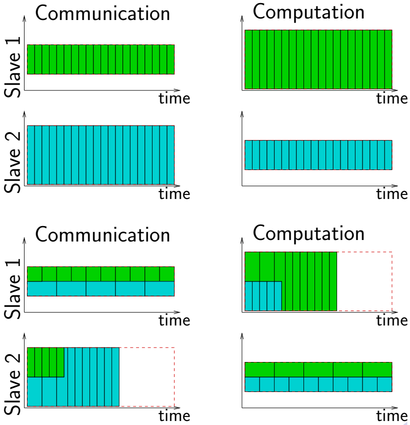
Even in systems consisting of independent elements, optimality cannot be determined on each independent subsystem!!!

Cooperative Approach:

Application 1 is processed exclusively on computer 1 and application 2 on computer 2. The respective throughput is $\alpha_1^{(\text{coop})} = \alpha_2^{(\text{coop})} = 1$.

Non-Cooperative Approach:

$$\alpha_1^{(nc)} = \alpha_2^{(nc)} = \frac{3}{4}$$



Characterizing the Nash Equilibrium

Theorem 8.

For a given system (N, B, W, K, b, w) there exists exactly one Nash Equilibrium and it can be analytically computed.

Proof.

Under the non-cooperative assumption, on a given worker, an application is either communication-saturated or computation-saturated.

Putting schedules in some canonical form enables to determine for each processor, which applications are communication-saturated and which ones are computation-saturated and then to derive the corresponding rates. □

Pareto Optimality

When is our Nash Equilibrium Pareto-optimal ?

Pareto Optimality

When is our Nash Equilibrium Pareto-optimal ?

Theorem 9.

The allocation at the Nash equilibrium is Pareto inefficient if and only if there exists two workers, namely n_1 and n_2 such that all applications are communication-saturated on n_1 and computation-saturated on n_2 (i.e. $\displaystyle\sum_k \frac{B_{n_1}}{W_{n_1}} \frac{w_k}{b_k} \leq K$ and $\displaystyle\sum_k \frac{b_k}{w_k} \frac{W_{n_2}}{B_{n_2}} \leq K$).

Corollary: on a single-processor system, the allocation at the Nash equilibrium is Pareto optimal.
Here Selfishness Degradation Factor is at least 2.

Braess-like Paradox

Pareto-inefficient equilibria can exhibit unexpected behavior.

Definition: **Braess Paradox**.

There is a Braess Paradox if there exists two systems ini and aug such that

$$ini < aug \text{ and } \alpha^{(nc)}(ini) > \alpha^{(nc)}(aug).$$

Braess-like Paradox

Pareto-inefficient equilibria can exhibit unexpected behavior.

Definition: **Braess Paradox**.

There is a Braess Paradox if there exists two systems ini and aug such that

$$ini < aug \text{ and } \alpha^{(nc)}(ini) > \alpha^{(nc)}(aug).$$

Theorem 10.

In the non-cooperative multi-port scheduling problem, Braess like paradoxes cannot occur.

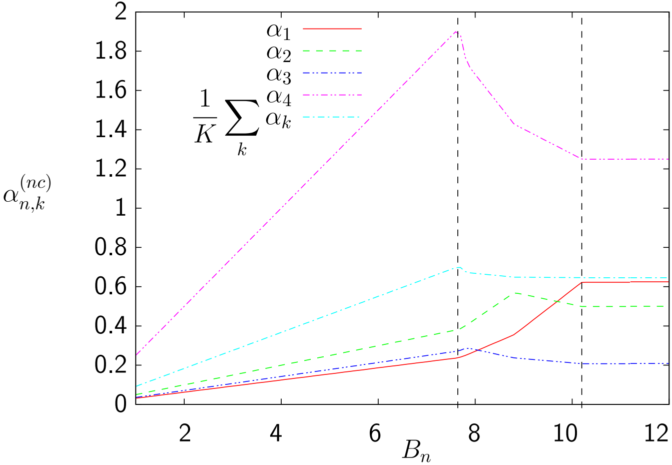
Proof.

- Defining an equivalence relation on sub-systems.
- Defining an order relation on equivalent sub-systems.

□

Pareto optimality and monotonicity of performance measures

Numerical example with a single slave and $K = 4$ applications.



Most classical performance measures decrease with resource augmentation!

Conclusion

Conclusion

- Applying fair and optimal sharing on each resource does not ensure any fairness nor efficiency when users do not cooperate.

  $\rightsquigarrow$ either applications cooperate or new complex and global access policies should be designed

Future Work

- Measuring Pareto-inefficiency is an open question under investigation.

Lecture's Summary

In one-shot, simultanous move, perfect information games:

- Equilibria (solution points, Nash Equilibria) are defined as being stable to users' selfish interests
- Pure strategies are equivalent to actions
- Mixed strategies are prob. distributions over the set of actions
- In mixed strategies, equilibria always exist for finite games (whether in zero-sum or not)
- In zero-sum games, equilibria (if they exist) are always unique

Additionally:

- Nash equilibria are generally not Pareto efficient
- One can define fairness criteria through a set of axioms or global objective function
- Fair points are unique (in convex set) and Pareto efficient
- A key issue for operators is to assess the efficiency of equilibria in systems they are managing to decide whether to meddle or not.