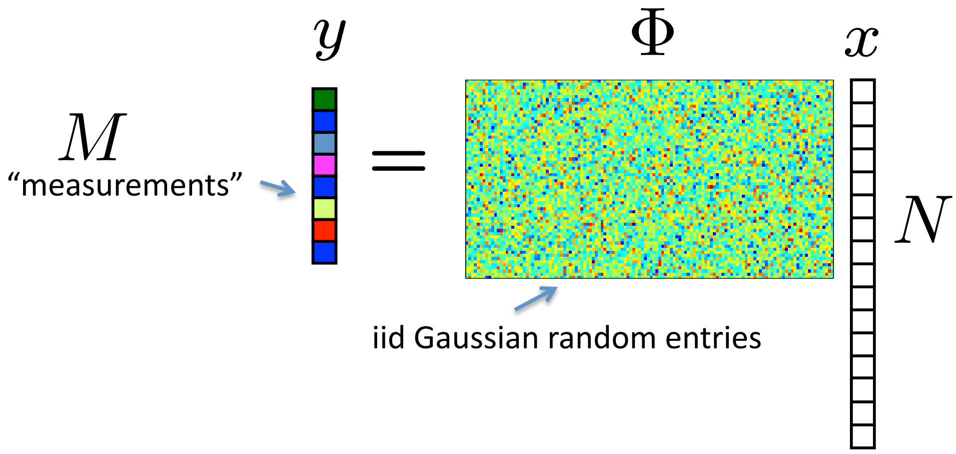
What kind of matrices are restricted isometries?

- They are very hard to design, but they exist everywhere!



- For *any fixed* $x \in \mathbb{R}^N$, we have

$$\mathrm{E}[\|\Phi x\|_2^2] = \|x\|_2^2$$

the mean of the measurement energy is exactly $\|x\|_2^2$

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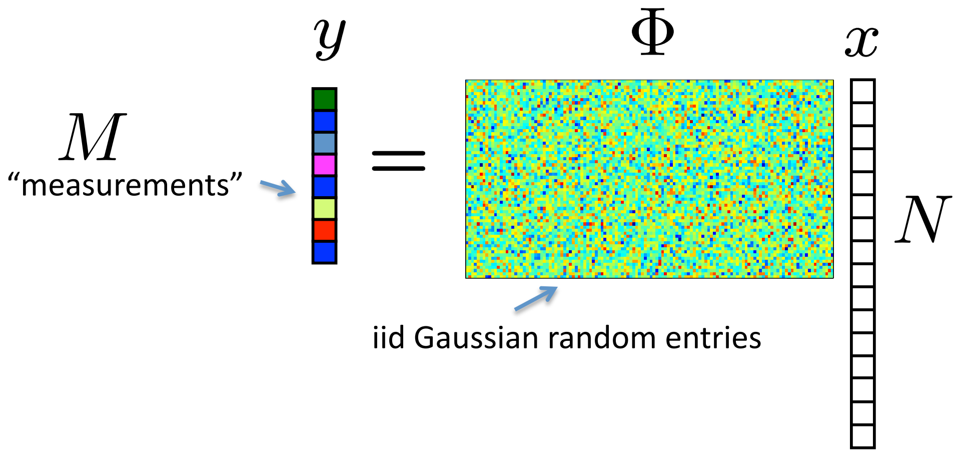

- For *any fixed* $x \in \mathbb{R}^N$, we have

$$\mathrm{P}\left\{\left|\|\Phi x\|_2^2 - \|x\|_2^2\right| < \delta\|x\|_2^2\right\} \;\geq\; 1 - e^{-M\delta^2/4}$$

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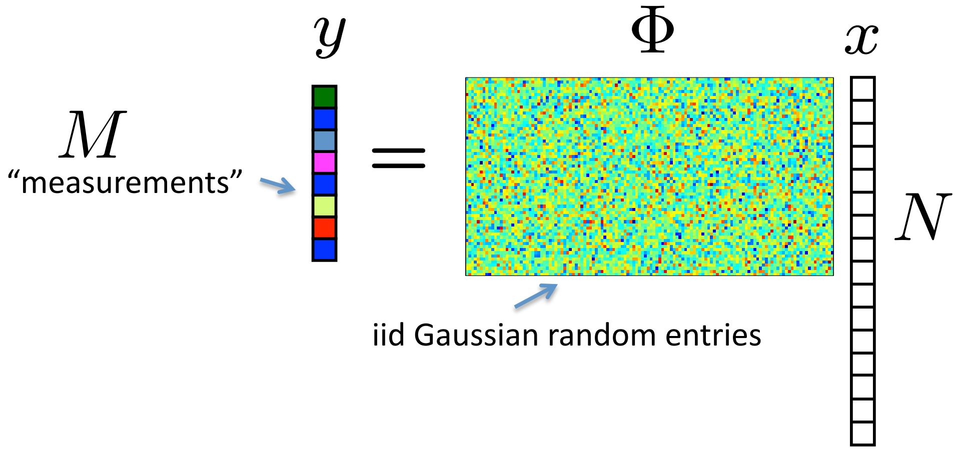


- For *all $2S$-sparse* $x \in \mathbb{R}^N$, we have

$$\mathrm{P}\left\{\max_x \left|\|\Phi x\|_2^2 - \|x\|_2^2\right| < \delta\|x\|_2^2\right\} \;\geq\; 1 - e^{c\cdot S\log(N/S)}e^{-M\delta^2/4}$$

So we can make this probability close to 1 by taking

$$M \gtrsim S\log(N/S)$$

What other types of matrices are restricted isometries?

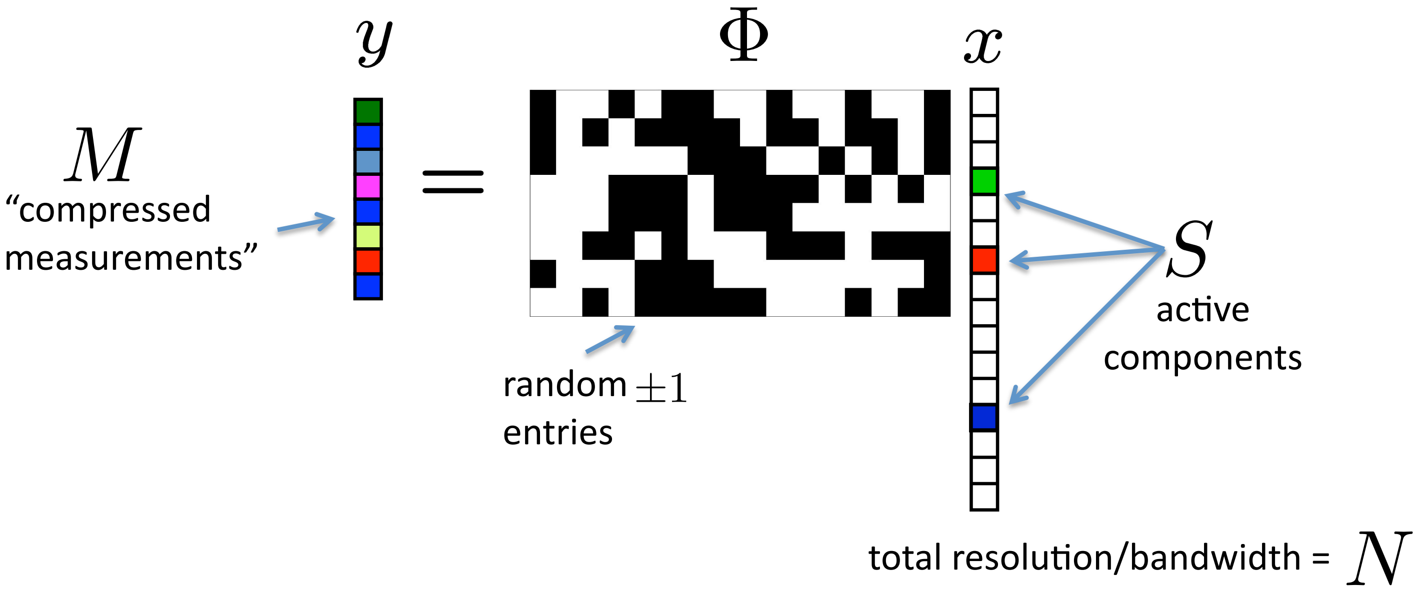
Four general frameworks:

- Random matrices (iid entries)
- Random subsampling
- Random convolution
- Randomly modulated integration

Note the role of randomness in all of these approaches

Slogan: *random projections keep sparse signal separated*

Random matrices (iid entries)

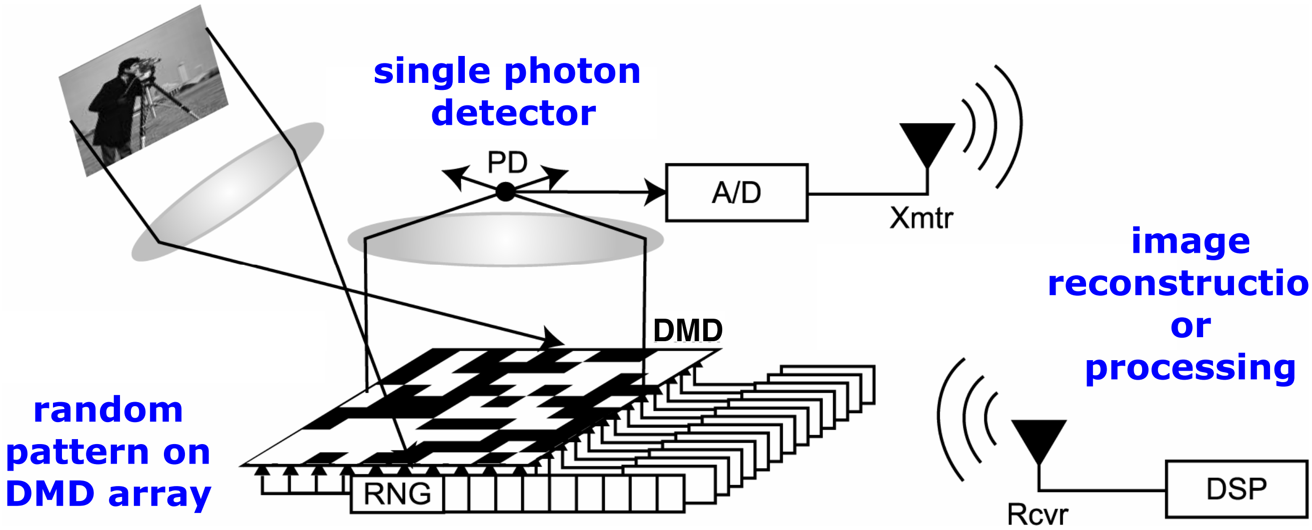


- *Random matrices* are provably efficient
- We can recover S-sparse x from

$$M \gtrsim S \cdot \log(N/S)$$

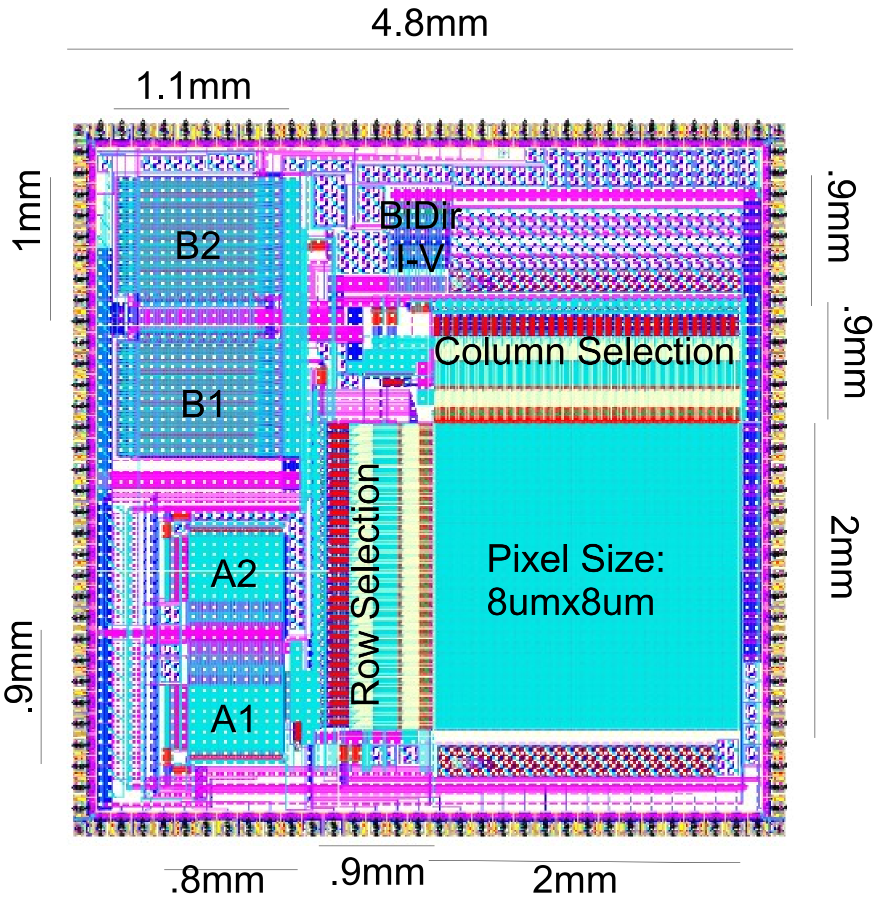
measurements

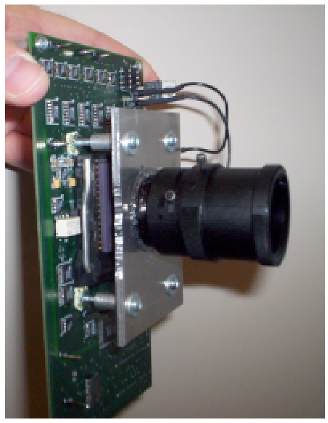
Rice single pixel camera



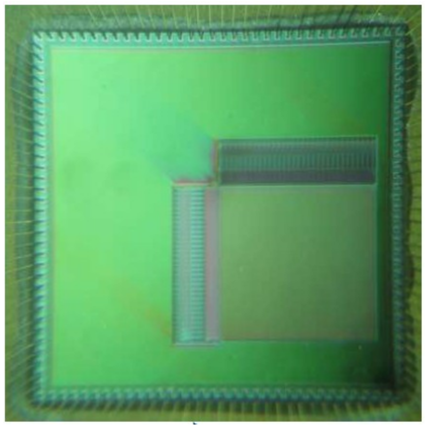
(Duarte, Davenport, Takhar, Laska, Sun, Kelly, Baraniuk '08)

Georgia Tech analog imager

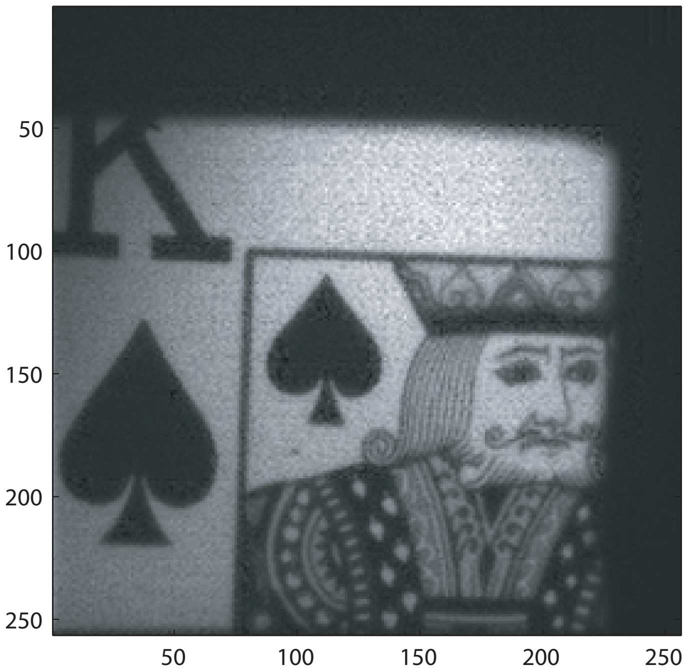




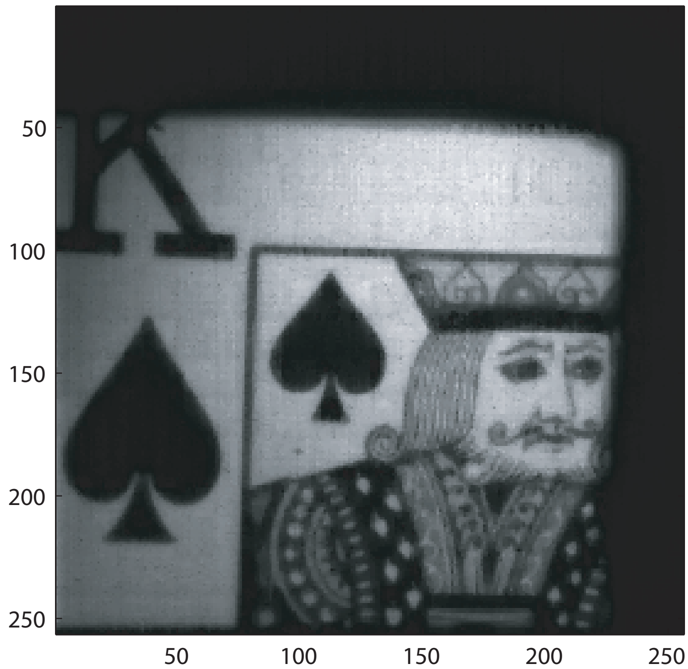
.35um CMOS process



Compressive sensing acquisition



10k DCT measurements

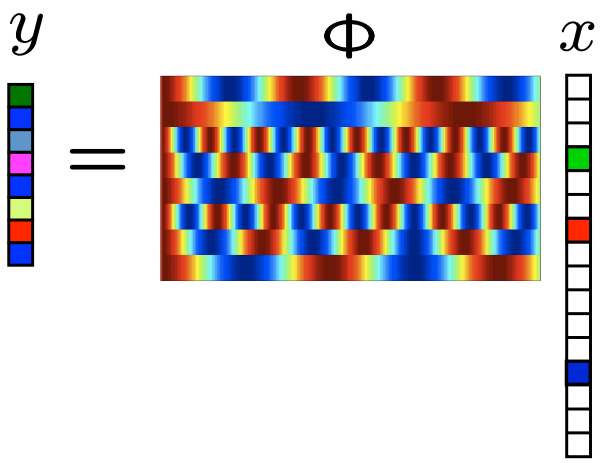


10k random measurements

(Robucci, Chiu, Gray, R, Hasler '09)

Random matrices

Example: Φ consists of *random rows* from an *orthobasis* U



Can recover S-sparse x from (Rudelson and Vershynin '06, Candès and R '07)

$$M \gtrsim \mu^2 \, S \cdot \log^4 N$$

measurements, where

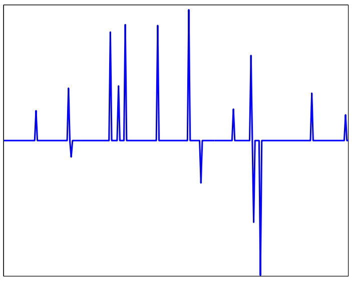
$$\mu = \sqrt{N} \max_{i,j} |(U^T \Psi)_{ij}|$$

is the *coherence*

Examples of incoherence

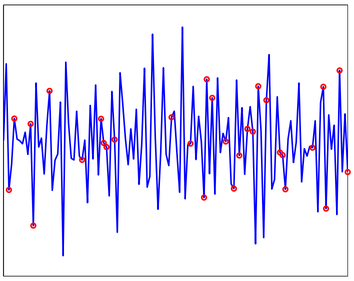
- Signal is sparse in time domain, sampled in Fourier domain

time domain $x(t)$



S nonzero components

freq domain $\hat{x}(\omega)$

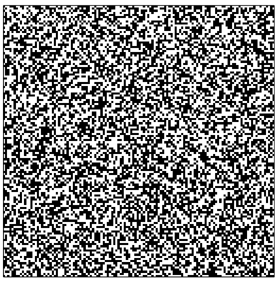


measure m samples

- Signal is sparse in wavelet domain, measured with noiselets

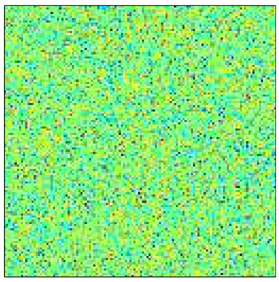
(Coifman et al '01)

example noiselet

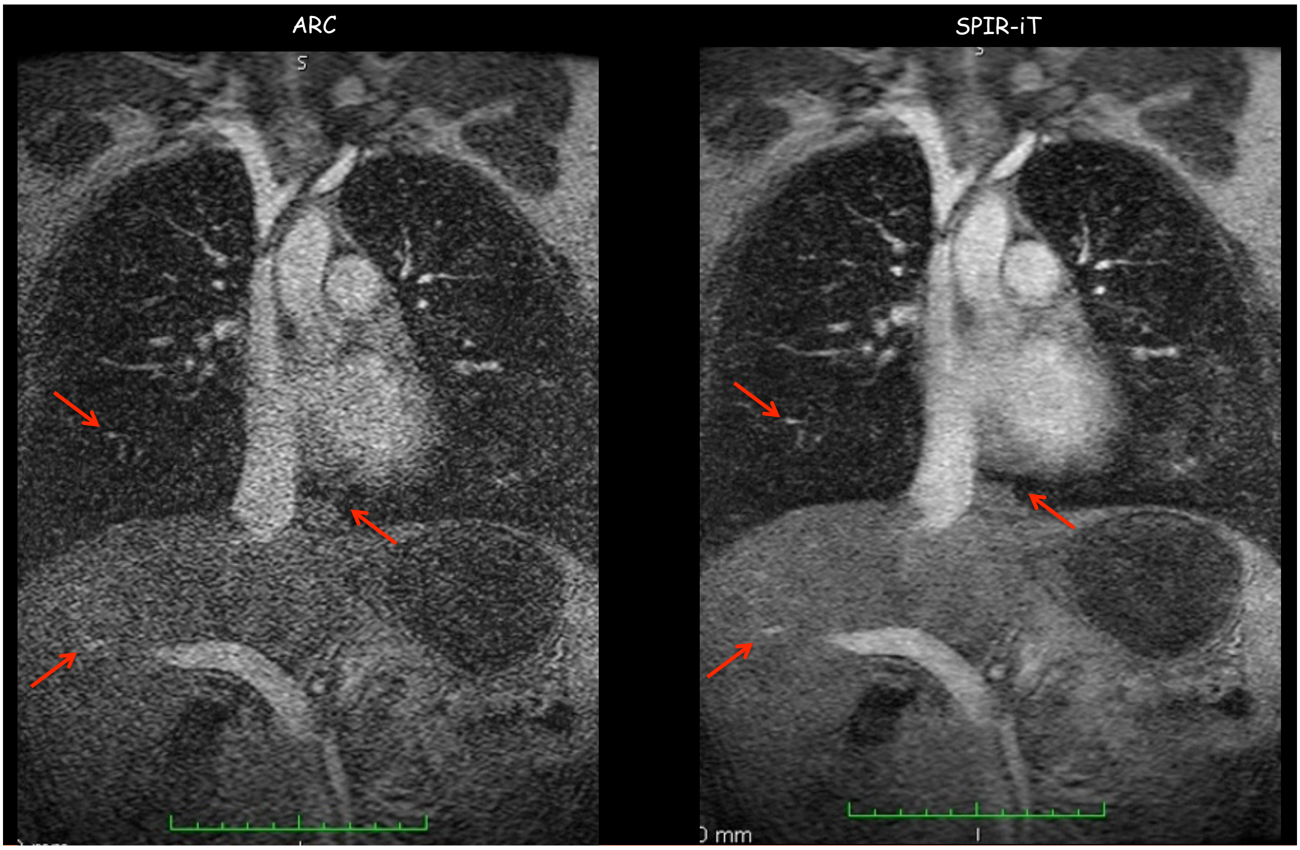


wavelet domain

noiselet domain



Accelerated MRI



(Lustig et al. '08)

Empirical processes and structured random matrices

- For matrices with this type of *structured randomness*, we simply do not have enough concentration to establish

$$(1-\delta)\|x\|_2^2 \;\leq\; \|\Phi x\|_2^2 \;\leq (1+\delta)\|x\|_2^2$$

  "the easy way"

- Re-write the RIP as a the *supremum of a random process*

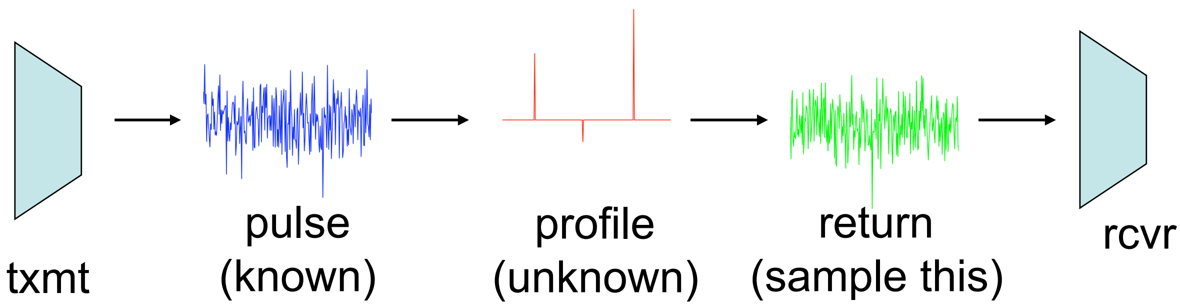
$$\sup_x |G(x)| = \sup_x |x^*\Phi^*\Phi x - x^* x| \;\;\leq\;\; \delta$$

  where the sup is taken over all $2S$-sparse signals

- Estimate this sup using tools from probability theory (e.g. the Dudley inequality) — approach pioneered by Rudelson and Vershynin

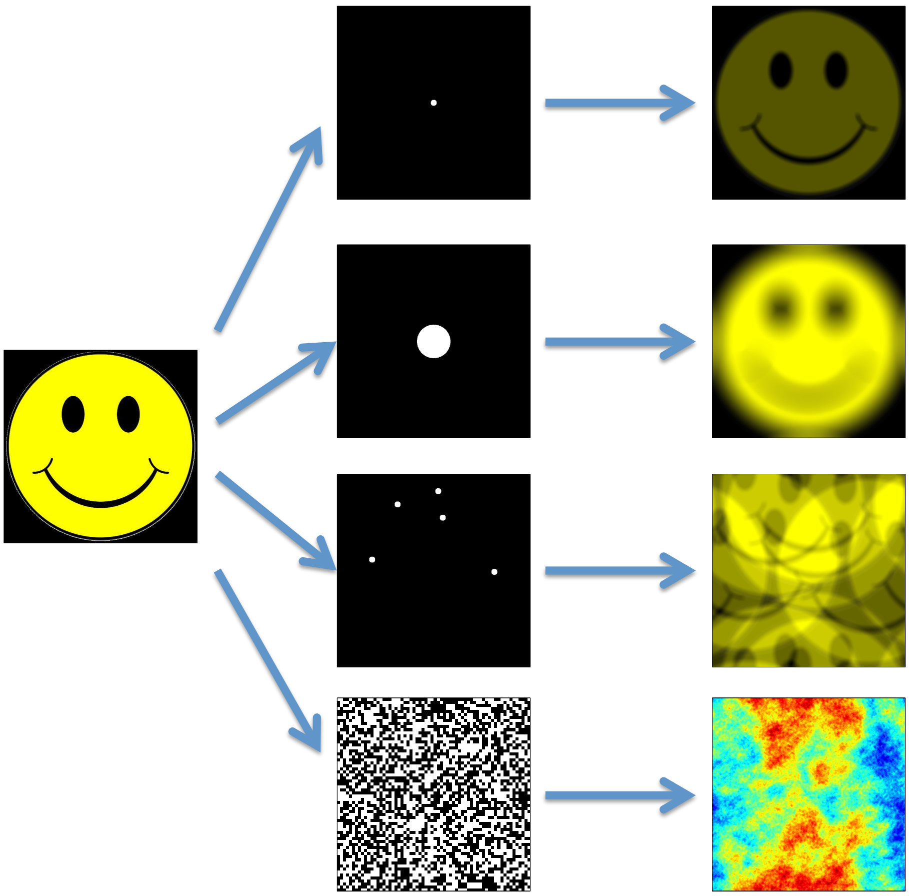
Random convolution

- Many *active imaging* systems measure a pulse convolved with a *reflectivity profile* (Green's function)



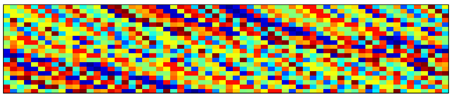
- Applications include:
  - radar imaging
  - sonar imaging
  - seismic exploration
  - channel estimation for communications
  - super-resolved imaging
- Using a *random pulse* = compressive sampling
  (Tropp et al. '06, R '08, Herman et al. '08, Haupt et al. '09, Rauhut '09)

Coded aperture imaging



Random convolution for CS, theory

- Signal model: sparsity in *any orthobasis* Ψ
- Acquisition model:
  generate a "pulse" whose FFT is a sequence of random phases (unit magnitude),
  convolve with signal,
  sample result at M *random* locations Ω



$$\Phi = R_\Omega \mathcal{F}^* \Sigma \mathcal{F}, \quad \Sigma = \mathrm{diag}(\{\sigma_\omega\})$$

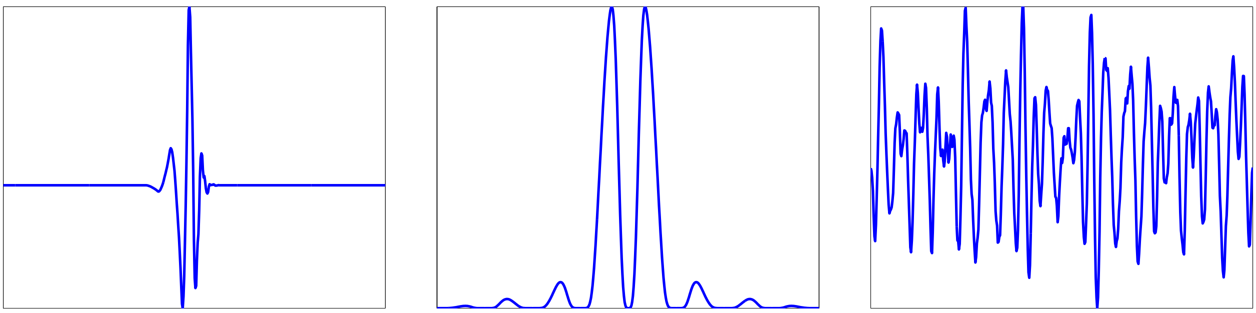
- The RIP holds for (R '08)

$$M \gtrsim S \log^5 N$$

  Note that this result is *universal*
- Both the random sampling and the flat Fourier transform are needed for universality

Randomizing the phase

local in time | local in freq | *not* local in M



sample here

Why is random convolution + subsampling universal?

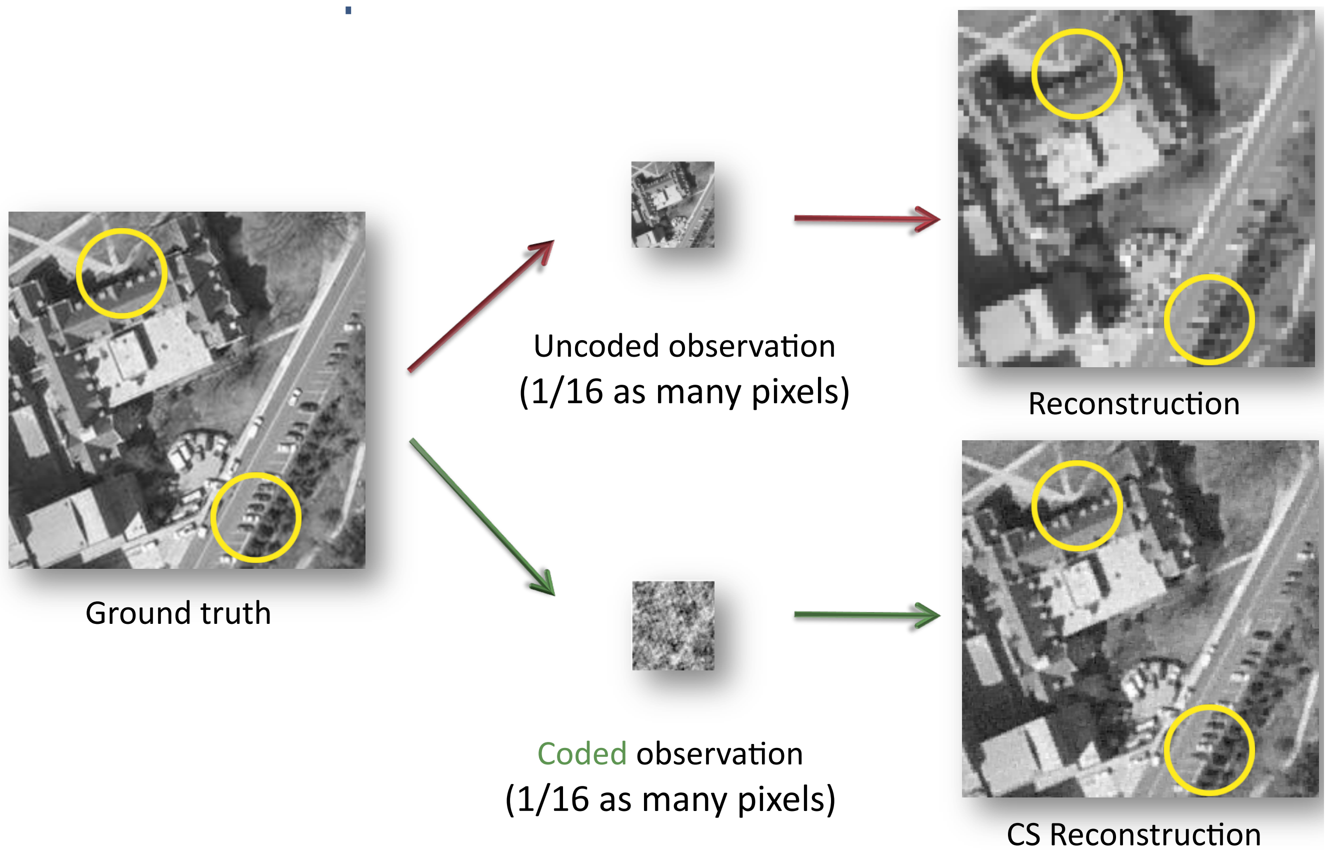
$$\left[\begin{array}{c} \\ \mathcal{F} \\ \\ \end{array}\right]\begin{bmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n\end{bmatrix}\left[\begin{array}{c|c|c|c} \hat{\psi}_1(\omega) & \hat{\psi}_2(\omega) & \cdots & \hat{\psi}_n(\omega) \\ & & & \\ \end{array}\right]$$

- One entry of $\Phi' = \Phi\hat{\Psi} = \mathcal{F}\Sigma\hat{\Psi}$:

$$\begin{aligned}\Phi'_{t,s} &= \sum_{\omega} e^{j2\pi\omega t}\sigma_\omega \hat{\psi}_s(\omega) \\ &= \sum_{\omega} \sigma'_\omega \hat{\psi}_s(\omega)\end{aligned}$$

- Size of each entry will be concentrated around $\|\hat{\psi}_s(\omega)\|_2 = 1$ *does not depend on the "shape" of* $\hat{\psi}_s(\omega)$

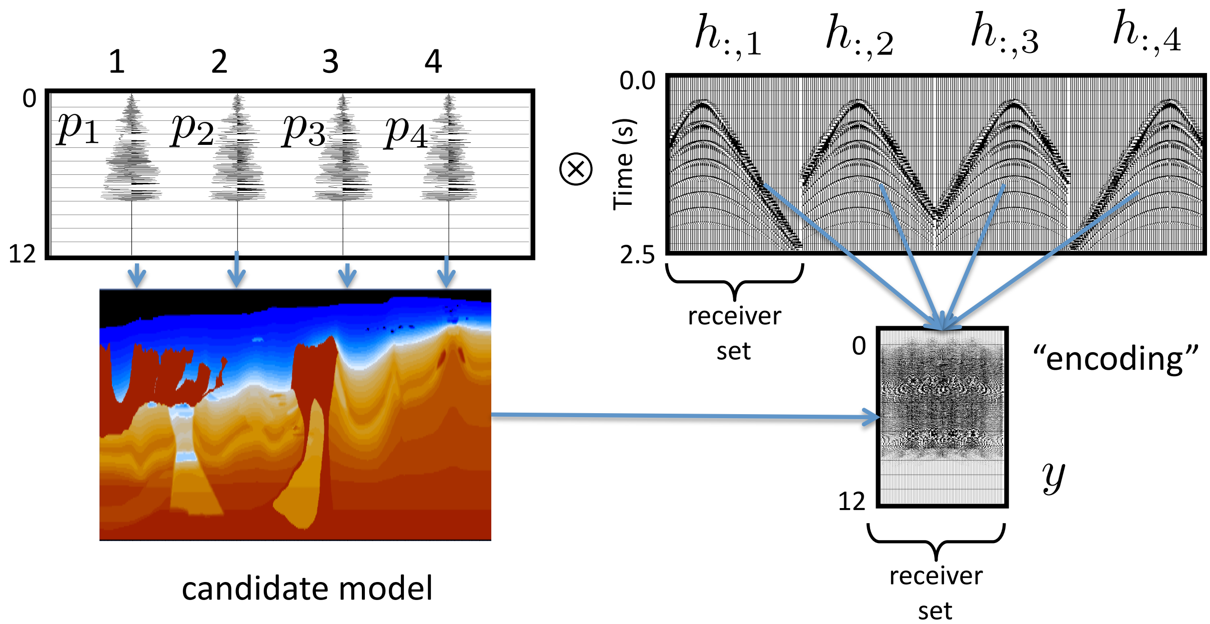
Super-resolved imaging



(Marcia and Willet '08)

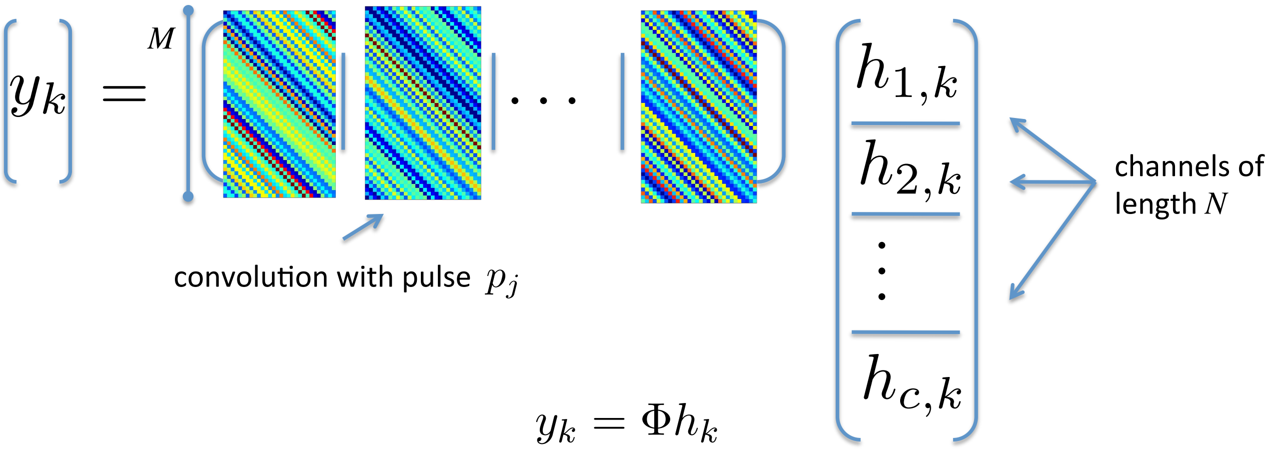
Seismic forward modeling

- Run a single simulation with all of the sources activated simultaneously with random waveforms
- The channel responses interfere with one another, but the randomness "codes" them in such a way that they can be separated later



Related work: Herrmann et. al '09

Restricted isometries for multichannel systems



- With each of the pulses as iid Gaussian sequences, Φ obeys

$$(1-\delta)\|h\|^2 \;\leq\; \|\Phi h\|_2^2 \;\leq\; (1+\delta)\|h\|_2^2 \quad \forall s\text{-sparse } h \in \mathbb{R}^{NC}$$

  when (R and Neelamani '09)

$$M \;\gtrsim\; S \cdot \log^5(NC) \;+\; N$$

- **Consequence:** we can separate the channels using short random pulses (using ℓ_1 min or other sparse recovery algorithms)