Introduction

Our "duality" results from the previous lecture are very good at characterizing the conditions under which a *particular* sparse x_0 is recoverable.

Unfortunately, there are two shortcomings:

1. The results are not uniform (i.e. they don't give guarantees for *all* sparse signals simultaneously)
2. They only apply to the "noise free" case (perfect sparsity, no noise)

In this lecture and the next, we will look at this problem from a different viewpoint that addresses both of these shortcomings.

Our goal

Theorem: Let Φ be an $M \times N$ matrix that is an approximate isometry for $3S$-sparse vectors. Let x_0 be an S-sparse vector, and suppose we observe $y = \Phi x_0$. Given y, the solution to

$$\min_x \ \|x\|_1 \quad \text{subject to} \quad \Phi x = y$$

is *exactly* x_0.

Moving to the solution

$$\min_x \; \|x\|_1 \quad \text{such that} \quad \Phi x = y$$

Call the solution to this $x^\sharp$. Set

$$h = x^\sharp - x_0.$$

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Two things must be true:

- $\Phi h = 0$
  Simply because both $x^\sharp$ and x_0 are feasible: $\Phi x^\sharp = y = \Phi x_0$
- $\|x_0 + h\|_1 \leq \|x_0\|_1$
  Simply because $x_0 + h = x^\sharp$, and $\|x^\sharp\|_1 \leq \|x_0\|_1$

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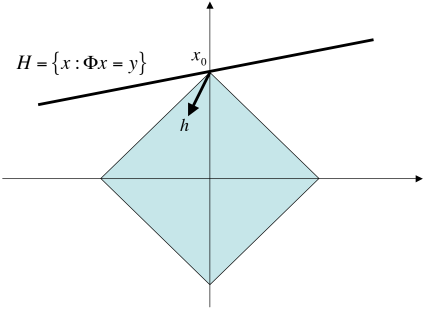
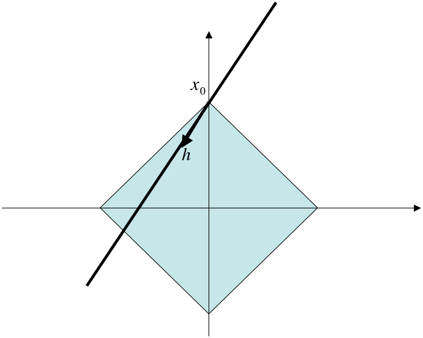
We'll show that if Φ is $3S$-RIP, then these conditions are *incompatible* unless $h = 0$

Geometry

SUCCESS



FAILURE



Two things must be true:

- $\Phi h = 0$
- $\|x_0 + h\|_1 \leq \|x_0\|_1$

Cone condition

For $\Gamma \subset \{1, \ldots, N\}$, define $h_\Gamma \in \mathbb{R}^N$ as

$$h_\Gamma(\gamma) = \begin{cases} h(\gamma) & \gamma \in \Gamma \\ 0 & \gamma \notin \Gamma \end{cases}$$

Let Γ_0 be the support of x_0. For any "descent vector" h, we have

$$\|h_{\Gamma_0^c}\|_1 \;\leq\; \|h_{\Gamma_0}\|_1$$

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Why? The triangle inequality..

$$\begin{aligned} \|x_0\|_1 \geq \|x_0 + h\|_1 &= \|x_0 + h_{\Gamma_0} + h_{\Gamma_0^c}\|_1 \\ &\geq \|x_0 + h_{\Gamma_0^c}\|_1 - \|h_{\Gamma_0}\|_1 \\ &= \|x_0\|_1 + \|h_{\Gamma_0^c}\|_1 - \|h_{\Gamma_0}\|_1 \end{aligned}$$

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Let Γ_0 be the support of x_0. For any "descent vector" h, we have

$$\|h_{\Gamma_0^c}\|_1 \;\leq\; \|h_{\Gamma_0}\|_1$$

We will show that if Φ is $3S$-RIP, then

$$\Phi h = 0 \quad \Rightarrow \quad \|h_{\Gamma_0}\|_1 \leq \rho \|h_{\Gamma_0^c}\|_1$$

for some $\rho < 1$, and so $h = 0$.

Some basic facts about ℓ_p norms

- $\|h_\Gamma\|_\infty \leq \|h_\Gamma\|_2 \leq \|h_\Gamma\|_1$
- $\|h_\Gamma\|_1 \leq \sqrt{|\Gamma|} \cdot \|h_\Gamma\|_2$
- $\|h_\Gamma\|_2 \leq \sqrt{|\Gamma|} \cdot \|h_\Gamma\|_\infty$

Dividing up $h_{\Gamma_0^c}$

Recall that Γ_0 is the support of x_0

Fix $h \in \mathrm{Null}(\Phi)$. Let

$$
\begin{aligned}
\Gamma_1 &= \text{locations of } 2S \text{ largest terms in } h_{\Gamma_0^c}, \\
\Gamma_2 &= \text{locations next } 2S \text{ largest terms in } h_{\Gamma_0^c}, \\
&\vdots
\end{aligned}
$$

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Then

$$0 = \|\Phi h\|_2 = \|\Phi(\sum_{j\geq 1} h_{\Gamma_j})\|_2 \geq \|\Phi(h_{\Gamma_0} + h_{\Gamma_1})\|_2 - \|\sum_{j\geq 2} \Phi h_{\Gamma_j}\|_2$$

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&\geq \|\Phi(h_{\Gamma_0} + h_{\Gamma_1})\|_2 - \sum_{j\geq 2} \|\Phi h_{\Gamma_j}\|_2
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Then

$$\|\Phi(h_{\Gamma_0} + h_{\Gamma_1})\|_2 \;\leq\; \sum_{j\geq 2} \|\Phi h_{\Gamma_j}\|_2$$

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\end{aligned}$$

Applying the $3S$-RIP gives

$$\begin{aligned}
\sqrt{1-\delta_{3S}}\|h_{\Gamma_0}+h_{\Gamma_1}\|_2 &\le \|\Phi(h_{\Gamma_0}+h_{\Gamma_1})\|_2 \\
&\le \sum_{j\ge 2}\|\Phi h_{\Gamma_j}\|_2 \;\le\; \sum_{j\ge 2}\sqrt{1+\delta_{2S}}\|h_{\Gamma_j}\|_2
\end{aligned}$$

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Applying the $3S$-RIP gives

$$\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \ \leq \ \sqrt{\frac{1+\delta_{2S}}{1-\delta_{3S}}} \sum_{j\geq 2} \|h_{\Gamma_j}\|_2$$

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Then

$$\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \;\le\; \sqrt{\frac{1+\delta_{2S}}{1-\delta_{3S}}} \sum_{j\ge 2} \sqrt{2S}\|h_{\Gamma_j}\|_\infty$$

since $\|h_{\Gamma_j}\|_2 \le \sqrt{2S}\|h_{\Gamma_j}\|_\infty$

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Then

$$
\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \;\leq\; \sqrt{\frac{1+\delta_{2S}}{1-\delta_{3S}}} \sum_{j\geq 1} \frac{1}{\sqrt{2S}} \|h_{\Gamma_j}\|_1
$$

since $\|h_{\Gamma_j}\|_\infty \leq \frac{1}{2S}\|h_{\Gamma_{j-1}}\|_1$

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Which means

$$\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \;\leq\; \sqrt{\frac{1+\delta_{2S}}{1-\delta_{3S}}}\,\frac{\|h_{\Gamma_0^c}\|_1}{\sqrt{2S}}$$

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Working to the left

$$\|h_{\Gamma_0}\|_2 \ \leq \ \|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \ \leq \ \sqrt{\frac{1+\delta_{2S}}{1-\delta_{3S}}} \, \frac{\|h_{\Gamma_0^c}\|_1}{\sqrt{2S}}$$

Dividing up $h_{\Gamma_0^c}$

Recall that Γ_0 is the support of x_0

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Working to the left

$$\frac{\|h_{\Gamma_0}\|_1}{\sqrt{S}} \;\leq\; \|h_{\Gamma_0}\|_2 \;\leq\; \|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \;\leq\; \sqrt{\frac{1+\delta_{2S}}{1-\delta_{3S}}}\,\frac{\|h_{\Gamma_0^c}\|_1}{\sqrt{2S}}$$

Wrapping it up

We have shown

$$\|h_{\Gamma_0}\|_1 \leq \sqrt{\frac{1+\delta_{2S}}{1-\delta_{3S}}}\sqrt{\frac{S}{2S}}\|h_{\Gamma_0^c}\|_1$$

$$= \rho\|h_{\Gamma_0^c}\|_1$$

for

$$\rho = \sqrt{\frac{1+\delta_{2S}}{2(1-\delta_{3S})}}$$

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We have shown

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for

$$\rho = \sqrt{\frac{1+\delta_{2S}}{2(1-\delta_{3S})}}$$

Taking $\delta_{2S} \leq \delta_{3S} < 1/3 \quad \Rightarrow \quad \rho < 1$.

SUCCESS!!

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