Asymptotic Behavior of Petri Nets

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Contents

Discrete Event Dynamic Systems

Discrete Event Dynamic Systems (DEDS)are essentially characterized by an countable state space E.

The instants when the system changes its state are called events and are countable (denoted $(T_n)_{n\in\mathbb{N}} \in \mathbb{R}$). Time is not discrete, but remarkable dates corresponding to the events are distinguished and form a countable set.
Time-driven systems: synchronous models
Event-driven systems: asynchronous models

Most man-made systems fall in either of these frameworks (computers, manufacturing systems, communication networks).

Evolution of a DEDS

If one looks at the evolution of a DEDS, at events, one gets:

$$\begin{aligned} x_{n+1} &= F_n(x_n, T_n, a_n, \xi_n) \\ T_{n+1} &= G_n(T_n, x_n, a_n, \xi_n) \end{aligned}$$

where $x_n, x_{n+1} \in E$ are the states of the systems at events n and $n+1$.

T_n is the instant of the n-th event.

$a_n, b_n \in A$ are actions of the controller (if the system is controlled).

$(\xi_n)_{n\in\mathbb{N}}$ is a random process, often called noise or perturbations.

Evolution of a DEDS (II)

$$\begin{aligned} x_{n+1} &= F_n(x_n, T_n, a_n, \xi_n) \\ T_{n+1} &= G_n(T_n, x_n, a_n, \xi_n) \end{aligned}$$

If F_n and G_n do not depend on n and on T_n, the system is called time homogeneous.

If F_n and G_n do not depend on ξ_n then the system is deterministic else, it is stochastic.

Problems to solve

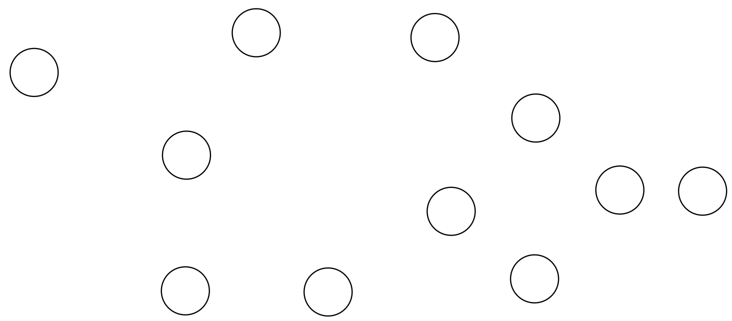
- Prove that T_n/n has a limit when $n \to \infty$. (First order limit) and compute it.

- Prove that x_n has a limit when $n \to \infty$. (in the stochastic case, $\Pr(x_n = i)$ has a limit when $n \to \infty$, called a stationary regime), and compute it. (Second order limits)

- Choose a_n, b_n such that the system satisfies some (asymptotic) property.

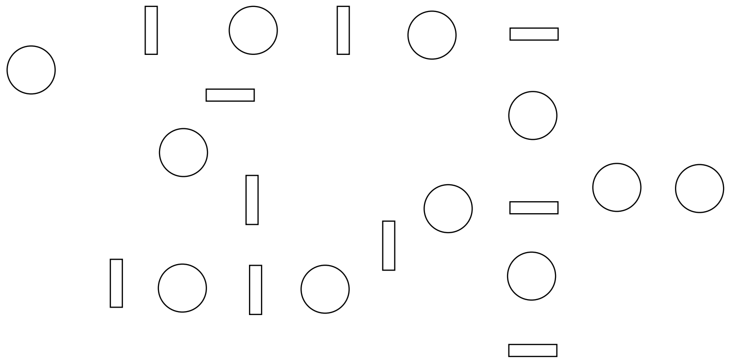
- Choose a_n, b_n to minimize some cost function, using the available information.
  no information: open-loop systems
  full information: close-loop systems

Petri Nets



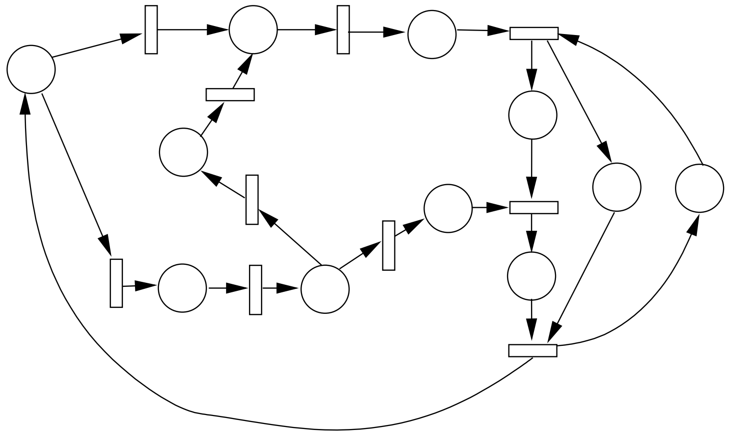
- Bipartite graph G (places P

Petri Nets



- Bipartite graph G (places P and transitions Q).

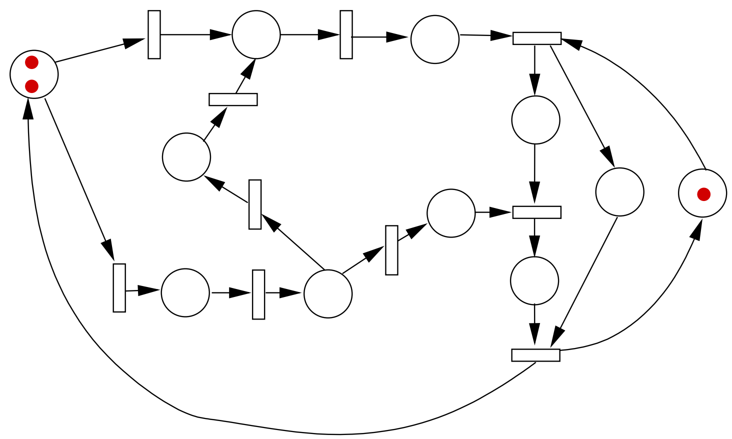
Petri Nets



- Bipartite graph G (places P and transitions Q).

- Arcs $C(p, q)$ and $C(q, p)$

Petri Nets

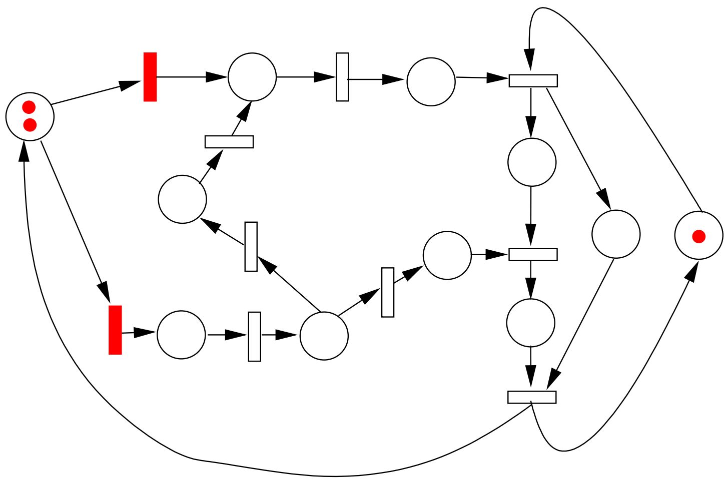


- Bipartite graph G (places P and transitions Q).

- Arcs $C(p, q)$ and $C(q, p)$

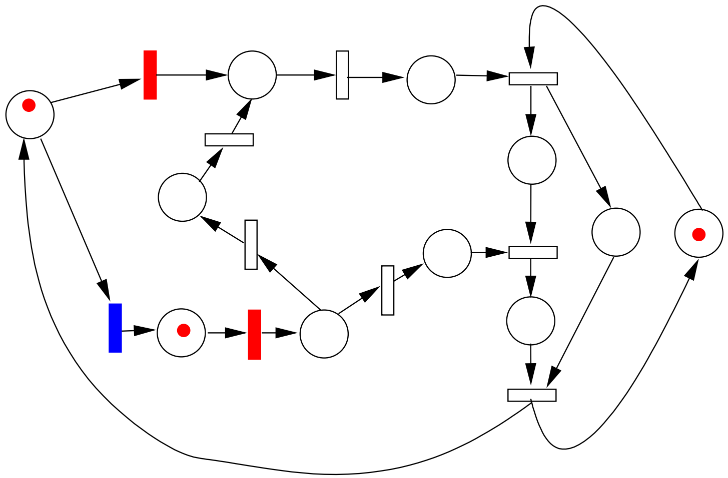
- Initial marking M_0 in places coding the current state.

Evolution of the marking: token game



A transition q is firable under marking M if $M \geqslant C(\cdot, q)$

Evolution of the marking: token game



A transition q is firable under marking M if $M \geqslant C(\cdot, q)$.

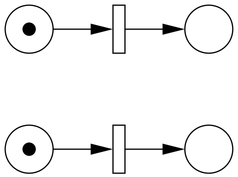
A firable transition can fire (an event) The marking becomes $M'(p) = M(p) - C(p,q) + C(q,p)$

Petri Nets: modelling power

Petri Nets is a classical tool to model Discrete Event Dynamic Systems.

Advantages: Graphical as well as a mathematical tool. Compact model (compared with the state space automaton).

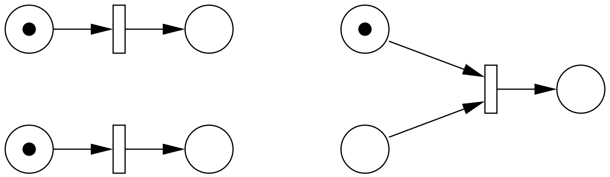
Modelling with Petri Nets



Natural model for distributed systems with

concurrency,

Modelling with Petri Nets



Natural model for distributed systems with

concurrency,

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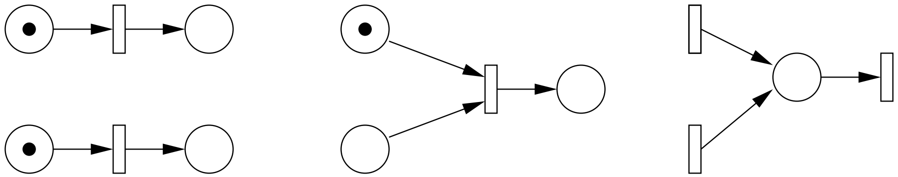
Modelling with Petri Nets



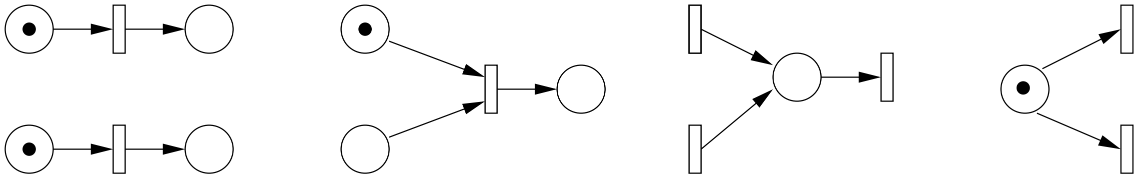
Natural model for distributed systems with

concurrency,

synchronizations,

asynchronous superpositions.

Modelling with Petri Nets



Natural model for distributed systems with

concurrency,

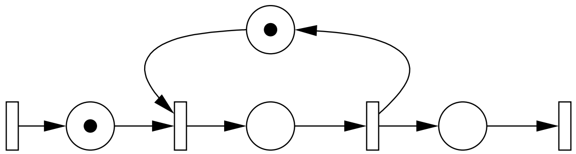
synchronizations,

asynchronous superpositions,

mutual exclusions.

Example 1: Critical section in railways

A section of a rail way

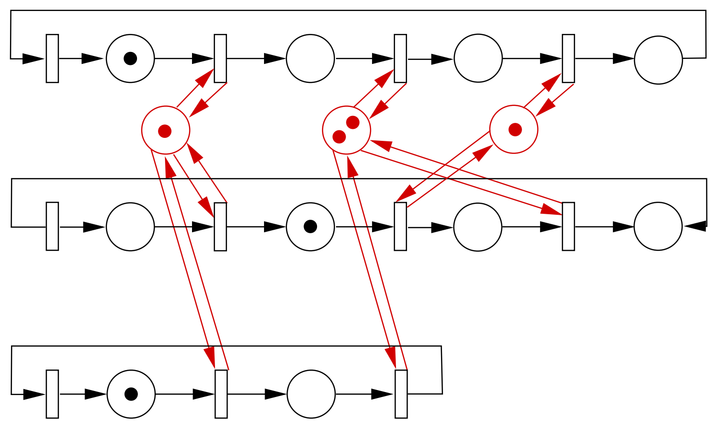


Critical section

Petri net models and the (max,plus) algebra is being used to model the futute high speed railway for Thalys in the Netherlands.

Manufacturing Systems: jobshops

N types of products are being manufactured by M different machines.



Product 1: M_1, M_2, M_3

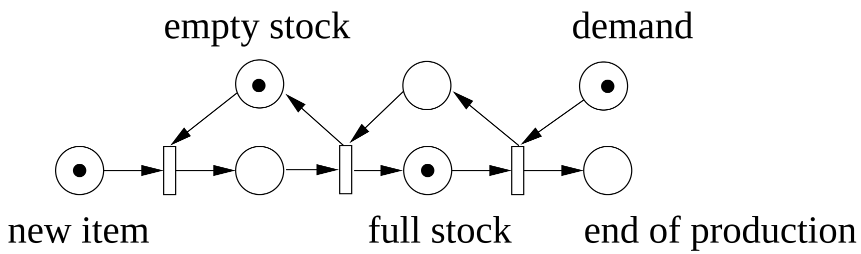
Product 2: M_1, M_3, M_2

Product 3: M_1, M_2

Petri nets are used to compute the schedule and the performance (identify the bottle neck). [Proth,Xie]

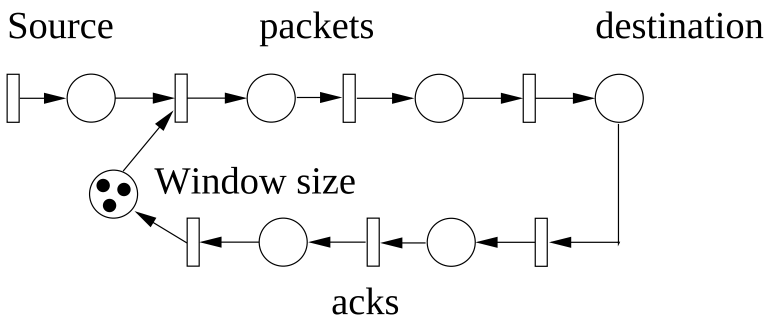
Manufacturing Systems: Kanban

Manufacturing chain avoiding stocks



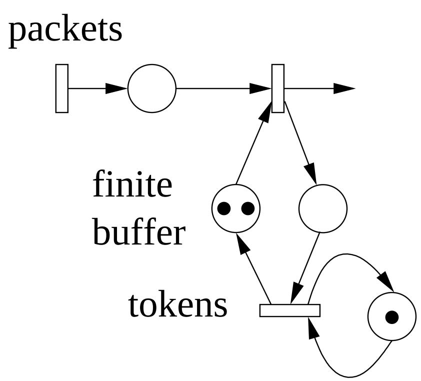
Petri nets were used to compute the performance of such systems. [Baccelli]

Communication networks: window control



[Baccelli,Bonald] Window control is used in the TCP protocol.

Communication networks: leaky buckets



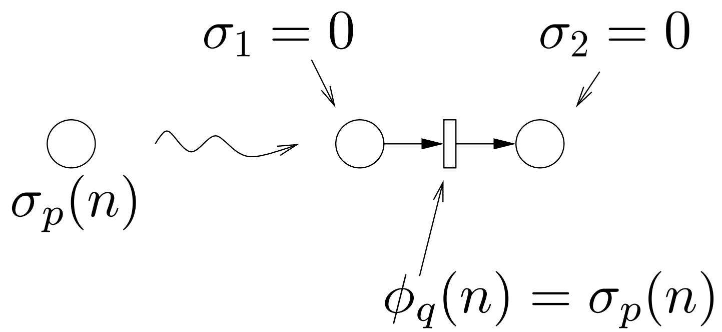
[Chang]

Timed Petri nets

$\sigma_p(n)$ is the n-th holding time in place p.

$\phi_q(n)$ is the n-th firing time of transition q.

With no loss of generality, we may assume that $\sigma_p(n) = 0$ for all places p, by using a local transformation.

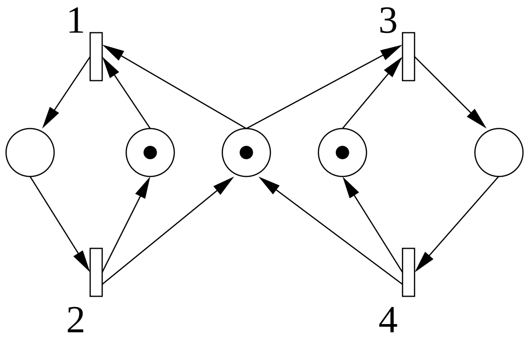


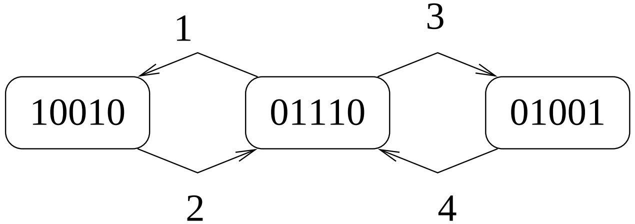
Markovian Approach: Reachability Graph

The reachability graph is an automata:

- Alphabet: Q (set of transitions)
- State space: all markings reachable from M_0
- Initial state: M_0
- Transitions: $M \xrightarrow{q} M'$ if firing q leads from M to M'.

Markovian Approach: Reachability Graph





Markovian Approach: exponential firing times

Assume that $\phi_q(n)$ is exponentially distributed and i.i.d.: $\Pr(\phi_q(n) > t) = e^{-\lambda_q t}$.

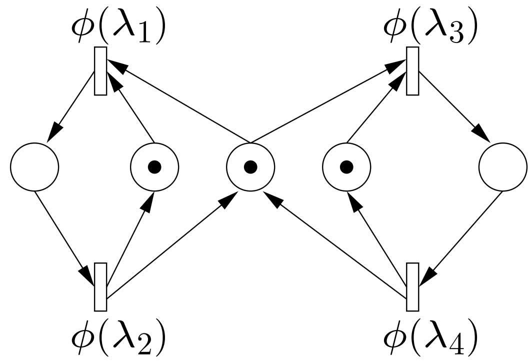
Assume that conflicts are solved using the *race policy*:

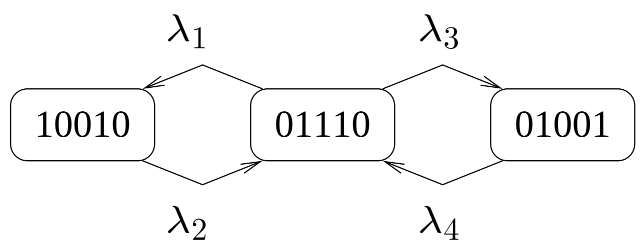
when two timed transitions compete for a token, they both initiate their firings.

The first one to finish gets the token. The firing of the other one aborts.

Theorem The reachability graph of an exponential Petri net under the race policy is a Continuous Time Markov Chain.

An example





Petri net with exponential firing times and the resulting infinitesimal generator.

Markovian Approach: Asymptotic Issues

The net reaches a stationary regime which can be computed by solve a linear system: $\pi Q = 0$, where Q is the infinitesimal generator matrix of the Markov Chain.

Problems:

State space may be infinite.

In the bounded case, the size of the automata is exponential in the size of the Petri net.

The exponential assumption is not valid in many cases. Even semi-markov generalizations are not always appropriate.

(max,plus): Event Graphs

Event graphs are Petri nets such that all the places have exactly on input and one output transition.

Event Graphs model systems involving synchronizations.

Event Graphs cannot model systems involving choices or asynchronous superposition.

(max,plus): State space

The usual state $M(t)$: marking at time t is replaced by the counters $N(t)$ or timers $X(n)$.

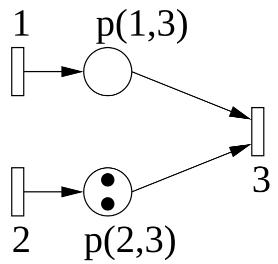
[Cohen, Quadrat, Viot]

Note that

$$M_{ij}(t) = N_i(t) - N_j(t),$$

$$M_{ij}(t) = \max\{n | X_i(n) \leqslant t\} - \max\{n | X_j(n) \leqslant t\}.$$

(max,plus):Evolution Equations



$$N_3(t) = \min \left(N_1(t - \sigma_{1,3} - \phi_1), N_2(t - \sigma_{2,3} - \phi_2) + 2 \right)$$

$$X_3(n) = \max \left(X_1(n) + \sigma_{1,3} + \phi_1, X_2(n-2) + \sigma_{2,3} + \phi_2 \right)$$

(max,plus): Algebra

$\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is an idempotent semi-ring with $-\infty$ as a null element and 0 as unity.

$\infty \oplus a = a$

$0 \otimes a = a$

$-\infty \otimes a = -\infty$

Idempotent: $a \oplus a = a$.

Matrix operations:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$$

$$(A \otimes B)_{ij} = \oplus_k A_{ik} \otimes B_{kj}$$

(max,plus): linear systems

The evolution equation of an event graph:

$$X_3(n) = \max\left(X_1(n) + \sigma_{1,3} + \phi_1, X_2(n-2) + \sigma_{2,3} + \phi_2\right)$$

can be rewritten

$$X_3(n) = \left(X_1(n) \otimes (\sigma_{1,3} + \phi_1)\right) \oplus \left(X_2(n-2) \otimes (\sigma_{2,3} + \phi_2)\right)$$

An event graph is a (max,plus) linear system. The evolution equation forms a Bellman Chain:

$$X(n+1) = X(n) \otimes A.$$

(max,plus):Spectral theorem

Theorem Let A be a (max,plus) irreducible matrix. then,

$$\exists n_0, c, \lambda, \quad \forall n \geqslant n_0, \quad A^{n+c} = c\lambda \otimes A^n$$

Corollary In a live, strongly connected event graph,

$$\forall n \geqslant n_0, \quad X(n+c) = c\lambda + X(n).$$

(max,plus): Computational issues

All the ingredients can be computed in polynomial time in the size of the system

cyclicity c: $O(N^2)$ Denardo

Cycle time λ: $O(N^3)$ Karp, $O(N)$ Howard

Coupling time: n_0 $O(N^k)$ Arguelles

(max,plus): Extensions

This approach can be extended to arbitrary one-bounded Petri nets using (max,plus) automata.

This has close links with heap of pieces ($\grave{a}$ la $Tetris$).

However, the computation of the cycle time is more difficult in the general case (open problem)

[Gaubert, Mairesse]

(max,plus): stochastic case

We assume that the firing times and the holding times are random variables

$$X(n+1) = X(n) \otimes A(n).$$

If the firing times and the holding times form stationary ergodic sequences. More precisely, $A(n) = A \circ \theta^n$, where θ is an ergodic flow in the probability space.

(max,plus): Kingman theorem

Theorem First order theorem

$$lim_{n\to\infty}\frac{X(n)}{n} = \gamma a.s,$$

where γ is the Lyapounov exponent of the system.

Proof Let $\xi_{m,m+k} = |A(m) \otimes \cdots \otimes A(m+k)|$,

$\xi_{m,m+k} \leqslant \xi_{m,m+p} + \xi_{m+p,m+k}$ is sub-additive.

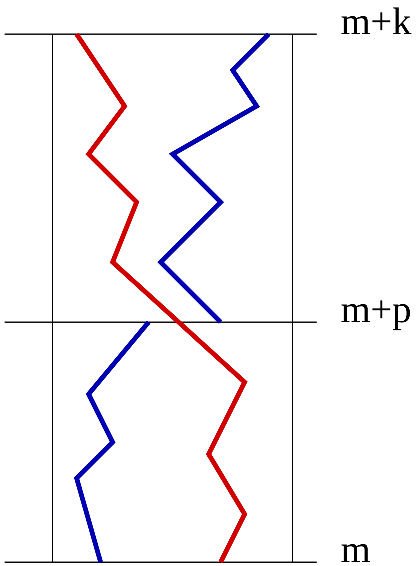
Kingman's sub-additive ergodic theorem concludes the proof.

Theorem Second order theorem

$lim_{n\to\infty}(X_i(n) - X_j(n))$ exists a.s.

[Baccelli, Foss, 95]

(max,plus):Sub-additivity



$$\xi_{m,m+k} \leqslant \xi_{m,m+p} + \xi_{m+p,m+k}$$

(max,plus): Computational issues

Approximate the Lyapounov exponent γ is NP-Hard.

[Gaubert, Blondel, 99]

For multidimensional Bernouilli cases, a development en series of γ can be computed (the size of the coefficients grows exponentially).

Computing the asymptotic regime is even harder.

The saturation rule: routed Petri nets

Stochastic Petri nets have conflicts which can be solved using routing functions:

$r_p(n)$ gives the transition where the nth token entering place p is routed to.

The routing is *weakly fair* if

$$\forall q, lim_{n \to \infty} \mathbf{1}_{r_p(n)=q} = \infty.$$

The saturation rule: Blocking transitions

Consider a live and bounded weakly fair routed Petri net (well-formed).

If an arbitrary transition is blocked after some time, then the evolution stops in finite time and the final state is unique.

[G. Mairesse, Haar]

The saturation rule: the monotone-separable framework

Consider an I-O discrete event system.

The input is a Marked Point Process I.

The input is a Marked Point Process O.

The system is given as an operator ϕ and $\phi(I) = O$. ϕ is monotone-separable if it is

Causal $\phi(X) \geqslant X$
Homogeneous $\phi(X + c) = \phi(X) + c$
Monotone: $X \leqslant Y \Rightarrow \phi(X) \leqslant \phi(Y)$
Separable: $\max(\phi(X)) \leqslant \min(Y) \Rightarrow \phi(X + Y) = \phi(X) + \phi(Y)$.

The saturation rule: Asymptotic results

A system satisfying the monotone-separable framework is non-expansive for the L_∞-norm. This implies to existence of a Lyapounov exponent γ_0.

The firing times and the holding times form unbounded joint stationary ergodic sequences.

The routing is made of an iid sequences.

$$\gamma_0 = \lim_{n\to\infty} \frac{1}{n} \text{ Last } (n \text{ at time } 0)$$

The saturation rule: Computing the firing rates

Let R be the routing matrix. Then the firing rates in the system λ satisfy $\lambda = \lambda R$.

This gives an easy way to compute λ up to a multiplicative constant.

Some open questions

1) minimal representation of a (max,plus) system.

2) Compute the Lyapounov exponent in interesting cases.

3) Investigate the cases where the dynamic is not non-expansive.