

Polyhedral Elimination Techniques

Pascaline Seminar, ENS Lyon, 04.12.2025

R. Mohr, joint (partially ongoing) work with A. Esterov and Y. Mukhina



Introduction

Lotka-Volterra equations [LYNCH 10]

$$\begin{cases} x_1' = 16x_1 - x_1x_2 \\ x_2' = -19x_2 + 19x_1x_2 \end{cases} \quad x_1(t), x_2(t) \text{ population densities}$$

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$$\begin{cases} y = x_1 \\ x_1' = -16x_1 - x_1x_2 \\ x_1'' = 95x_1x_2 + 57x_1 + 196 \end{cases}$$

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Compute Newton polytope of $f_{\min} \rightarrow$ point sampling $\rightarrow$ interpolation of $f_{\min}$

Given

Complete intersection

$$X = \mathbf{V}(f_0, \dots, f_k) \subset \mathbb{C}_{\mathbf{y}}^{n-k} \times \mathbb{C}_{\mathbf{x}}^k$$

$$\pi : \mathbb{C}_{\mathbf{y}}^{n-k} \times \mathbb{C}_{\mathbf{x}}^k \rightarrow \mathbb{C}_{\mathbf{y}}^{n-k}$$

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Evaluation-interpolation via

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This talk

Compute Newton polytope of g
from those of $f_0, \dots, f_k$

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Theorem [ESTEROV, KHOVANSKII 08]

$\Delta_0, \dots, \Delta_k$ NP's of $f_0, \dots, f_k$, $\Delta(g)$ NP of g . Then:

$$\Delta(g) \subseteq \text{MF}_{\pi}(\Delta_0, \dots, \Delta_k),$$

the mixed fiber polytope of $\Delta_0, \dots, \Delta_k$ with equality for generic f_i .

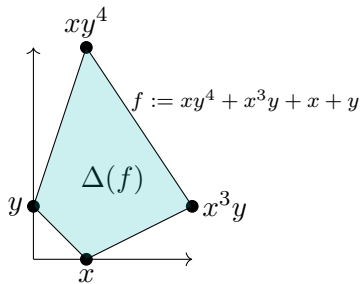
Tools from Polyhedral Geometry

Definition

A **polytope** is the convex hull of finitely many points in $\mathbb{R}^n$.

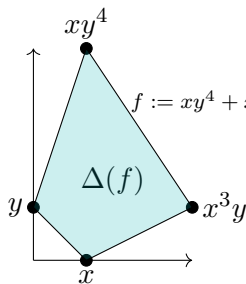
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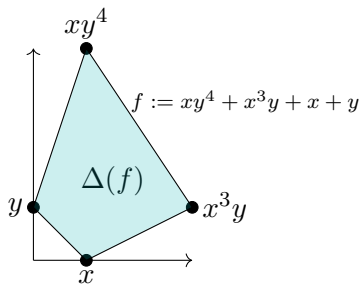
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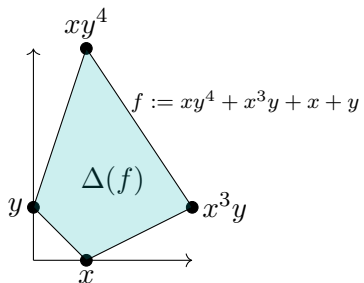


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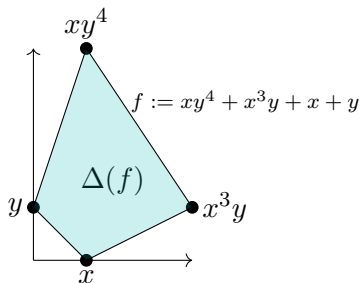


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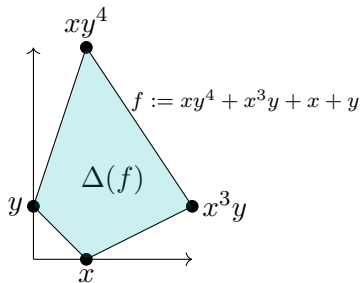
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Vertex oracle: Black box function with **input** $\gamma \in \mathbb{R}^n$ and **output**

$$p_0 \in \Delta \text{ s.t. } \gamma \cdot p_0 = \min \{ \gamma \cdot p \mid p \in \Delta \}.$$

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[HUGGINS 06]

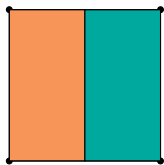
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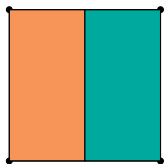
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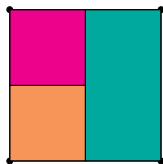
Subdivision S_1

Definition

Subdivision of polytope Δ : Writing Δ as union of smaller polytopes only touching in faces of themselves.

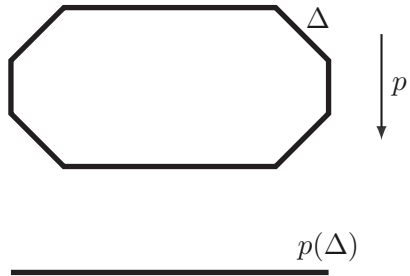


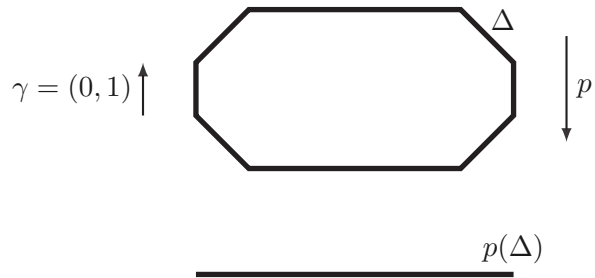
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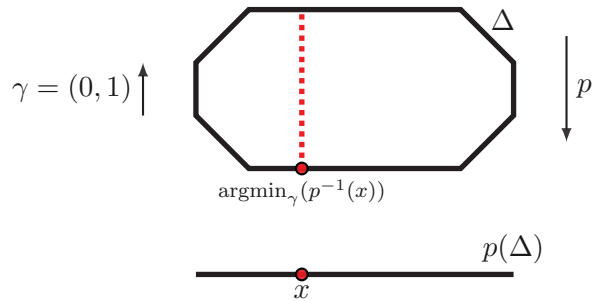


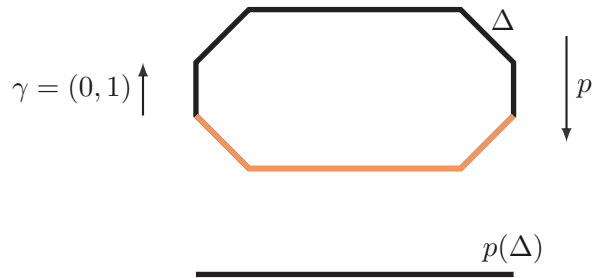
Subdivision S_2

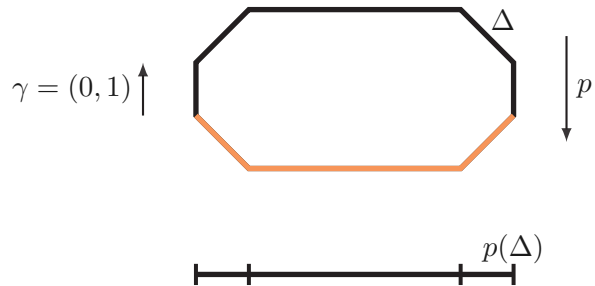
Subdivision S_2 **refines** subdivision S_1

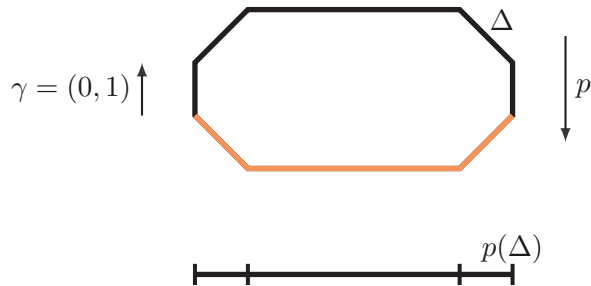




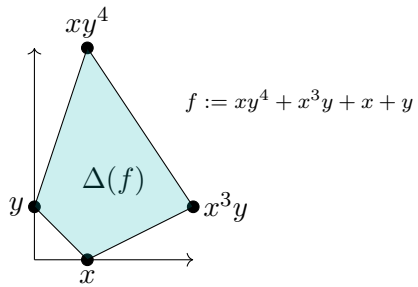


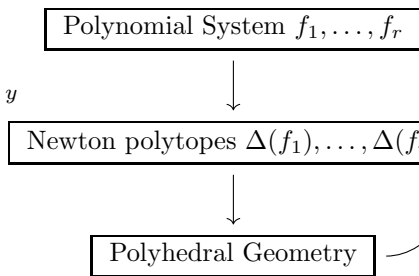
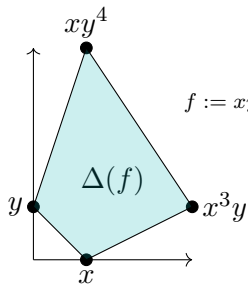




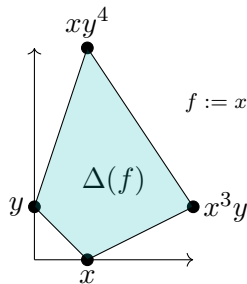


This construction: *p-coherent subdivision* of $p(\Delta)$.





Toric geometry
[Cox, Little, Schenck 11]



Polynomial System $f_1, \dots, f_r$



Newton polytopes $\Delta(f_1), \dots, \Delta(f_r)$



Polyhedral Geometry

Toric geometry

[Cox, LITTLE, SCHENCK 11]

Theorem [BERNSTEIN, KUSHNIRENKO, KHOVANSKII 75, 76, 95]

For $F = (f_1, \dots, f_n) \subset \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ we have

$$\# \text{ isolated roots of } F \text{ in } (\mathbb{C}^*)^n \leq n! \cdot \text{MV}(\Delta(f_1), \dots, \Delta(f_n)),$$

the **mixed volume**, with “=” for **generic** F .

Example

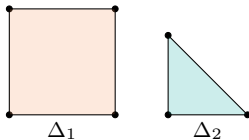
For $\Delta_1, \Delta_2 \in \mathbb{R}^2$, $\lambda_1, \lambda_2 \in \mathbb{R}$:

$$\text{Vol}(\lambda_1 \Delta_1 + \lambda_2 \Delta_2) = \lambda_1^2 \text{Vol}(\Delta_1) + 2\lambda_1 \lambda_2 \text{MV}(\Delta_1, \Delta_2) + \lambda_2^2 \text{Vol}(\Delta_2).$$

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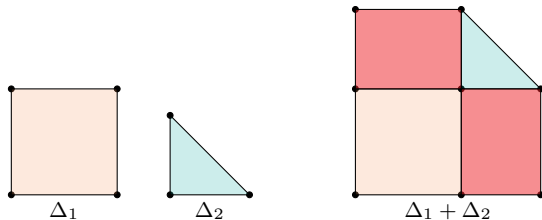
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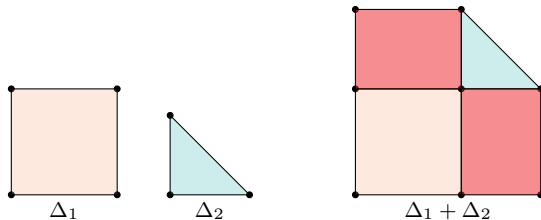
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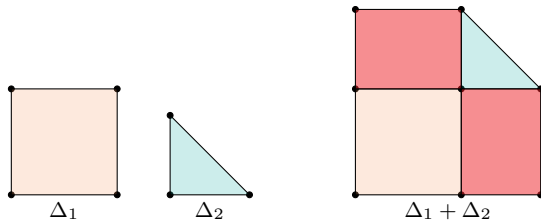


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Computation via **mixed subdivisions** (= $\blacksquare$) [HUBER, STURMFELS 95]

Mixed Fiber Polytopes

Given $X = \mathbf{V}(f_0, \dots, f_k) \subset \mathbb{C}_{\mathbf{y}}^{n-k} \times \mathbb{C}_{\mathbf{x}}^n$
 $\pi : \mathbb{C}_{\mathbf{y}}^{n-k} \times \mathbb{C}_{\mathbf{x}}^n \rightarrow \mathbb{C}_{\mathbf{y}}^n$

such that $\pi(X) = \mathbf{V}(g)$

Compute Newton polyope $\Delta(g) \subset \mathbb{R}^{n-k}$ from $\Delta(f_0), \dots, \Delta(f_k)$

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Proposition

Can choose g so that

$$\#\mathbf{V}(g, h_1, \dots, h_{n-k-1}) = \#\mathbf{V}(f_0, \dots, f_k, h_1, \dots, h_{n-k-1}).$$

for general polynomials $h_1, \dots, h_{n-k-1} \in \mathbb{C}[\mathbf{y}^{\pm}]$.

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Theorem [ESTEROV, KHOVANSKII 08]

$\Delta(g)$ up to affine shift, **uniquely determined** by

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for all convex bodies $P_1, \dots, P_{n-k-1} \subset \mathbb{R}^k$.

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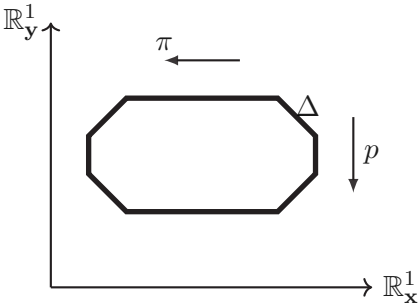
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Implicit definition of **mixed fiber polytope** $\text{MF}_{\pi}(\Delta(f_0), \dots, \Delta(f_k))$.

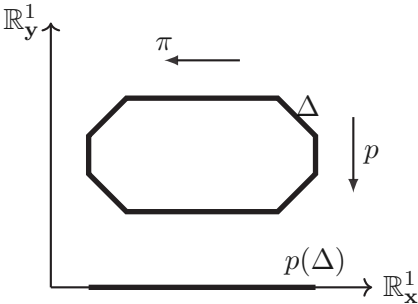
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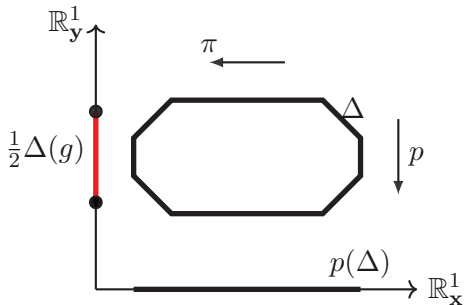
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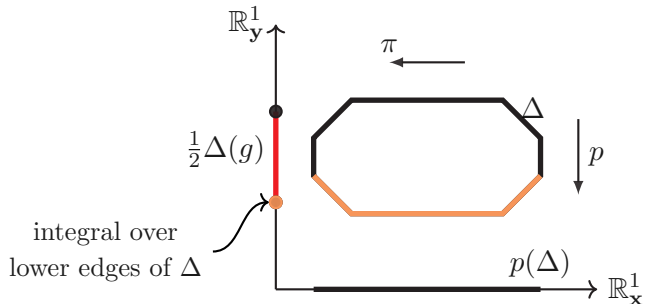


Definition [BILLERA, STURMFELS 92]

$1/2 \cdot \Delta(g)$ is the **fiber polytope** of Δ with respect to π :

$$\frac{1}{2} \Delta(g) = F_{\pi}(\Delta) := \int_{p(\Delta)} p^{-1}(t) \cap \Delta dt.$$

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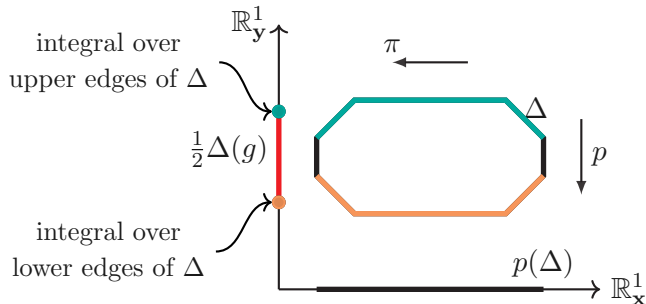


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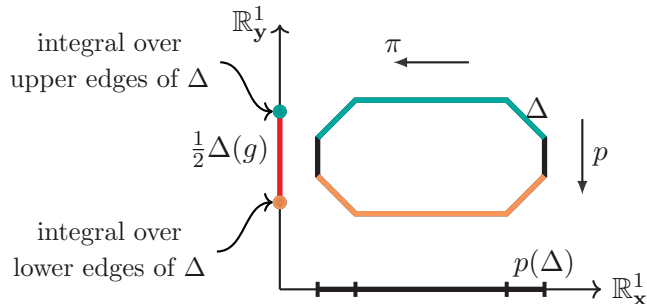


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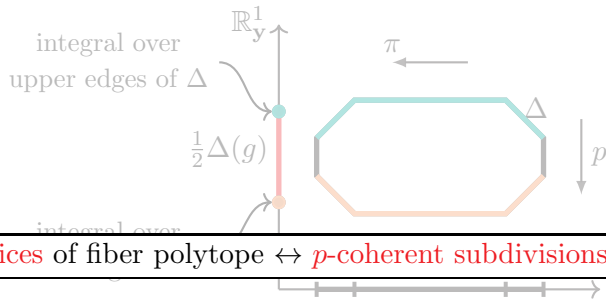


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Vertices of fiber polytope $\leftrightarrow$ p -coherent subdivisions of $p(\Delta)$

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The **mixed fiber polytope** $\text{MF}_\pi(\Delta_0, \dots, \Delta_k)$ is $1/(k+1)!$ the coefficient of $\lambda_0 \dots \lambda_k$ in $F_\pi(\lambda_0 \Delta_0 + \dots + \lambda_k \Delta_k)$.

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Core idea of our algorithm:

Input: Newton polytopes $\Delta(f_0), \dots, \Delta(f_k) \in \mathbb{R}_{\mathbf{y}}^k \times \mathbb{R}_{\mathbf{x}}^n$, $\gamma \in \mathbb{R}_{\mathbf{y}}^{n-k}$

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S mixed subdivision $\Rightarrow$

$$v_\gamma = \sum_{C \in S} \text{Vol}(C) \cdot \text{point}_C$$

Benchmarks (`julia` implementation)

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Reason: For $\text{trop}(X)$ computations with **normal fan** of $\Delta(f_0) + \dots + \Delta(f_k)$
 $\leadsto \# \text{ faces} = O(2^n)$

Differential dynamical system [LYNCH 10]

$$\begin{cases} x_1' = x_1^2 + x_1x_2 + x_2^2 + 1 \\ x_2' = x_2 \end{cases}$$

Compute **minimal** relation $f_{\min}(x_1, x_1', x_1'') = 0$ [MUKHINA, POGUDIN 25]

Differential dynamical system [LYNCH 10] $\rightsquigarrow$ Reduction to polynomial system

$$\left\{ \begin{array}{l} y = x_1 \\ x_1' = x_1^2 + x_1 x_2 + x_2^2 + 1 \\ x_1'' = x_1'(2x_1 + x_2) + x_2(2x_2 + 1) \end{array} \right.$$

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R. Mohr, Y. Mukhina – Computing Newton Polytopes of Eliminants –
Proceedings of ISSAC '25

Elimination for Engineered Complete Intersections

joint (ongoing) work with A. Esterov and Y. Mukhina

Recall:

Theorem [ESTEROV, KHOVANSKII 08]

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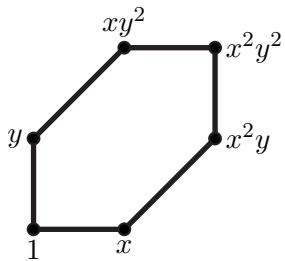
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Definition

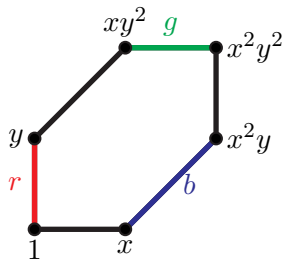
Engineered complete intersection (ECI): polynomial system of the form

$$\sum_{a \in A} v_{i,a} c_a \mathbf{x}^a = 0 \quad i = 0, \dots, k < n$$

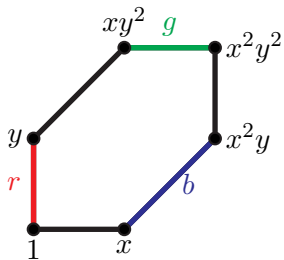
for $A \subset \mathbb{Z}^n$, **fixed** $v_{i,a}$ and **generic** c_a .



$$A = \{1, x, y, x^2y, xy^2, x^2y^2\}$$

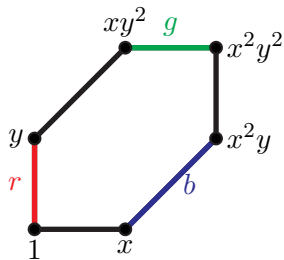


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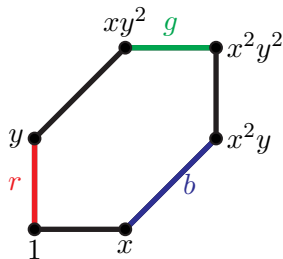
$$\text{ECI: } r + g + b = r + 2g + 3b = 0$$



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⚠ **Note:** generic root count is 3, but BKK predicts 6!

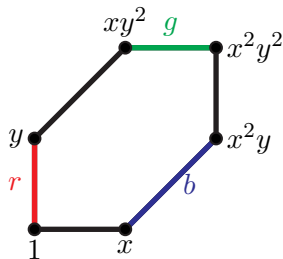


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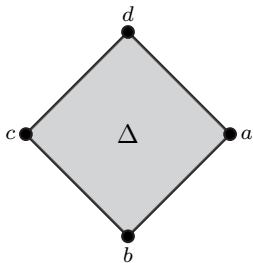
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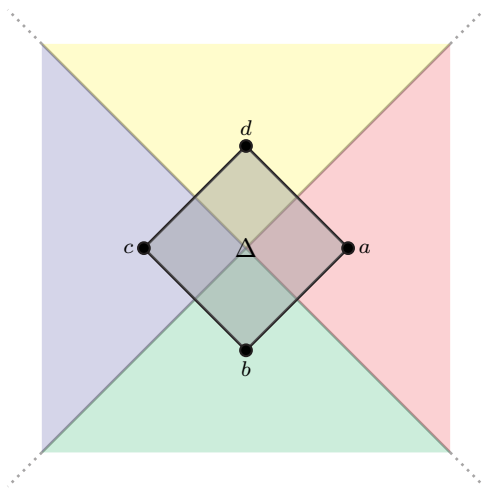
For elimination: What is a **mixed subdivision** for ECI's?

Support function of polytope Δ : $m_{\Delta}(\gamma) := \min \{ \gamma \cdot p \mid p \in \Delta \}$

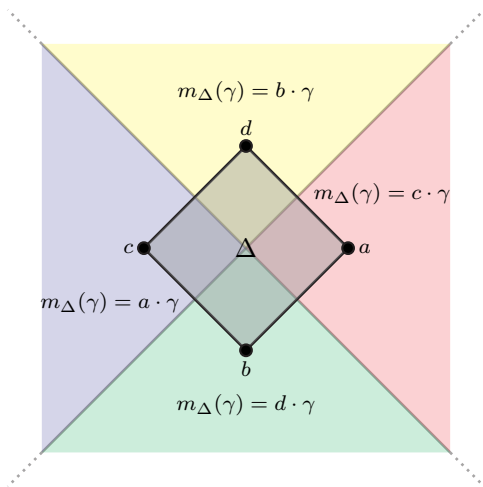
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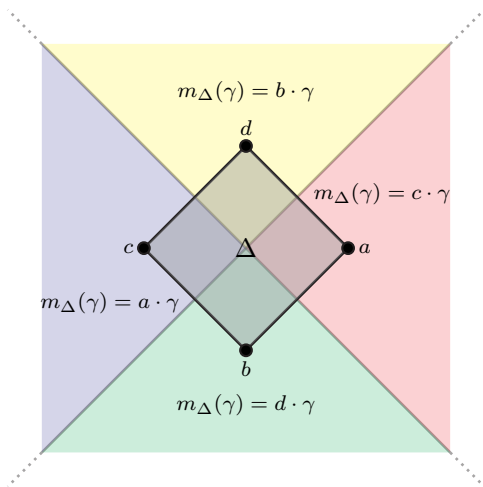
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m_{Δ} is a **conewise linear, continuous** function $\leftrightarrow$ the polytope Δ

Polytopes in theory of
sparse generic systems

[ESTEROV 24]

Piecewise linear
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Polytopes in theory of
sparse generic systems

[ESTEROV 24]

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functions of ECI's

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Theorem [ESTEROV 24]

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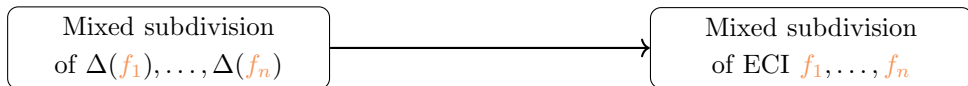
$$\text{MV}(m_1, \dots, m_n) = \sum_{C \in S} \text{Vol}(C).$$

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Input ECI $f_1, \dots, f_n$

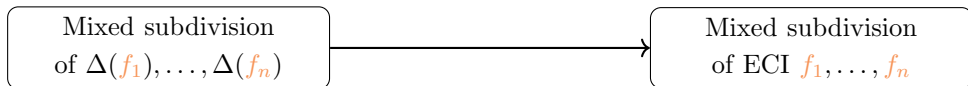
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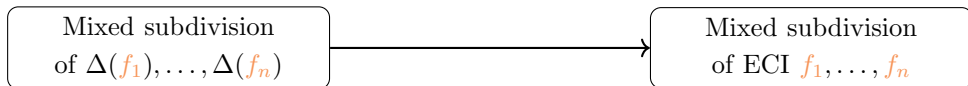
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Tropical analogue of homotopy continuation adapted from
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Thank you!