Recursive versions of the Levenberg-Marquardt reassigned spectrogram and of the synchrosqueezed STFT

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Outline

STFT implemented as a recursive filtering

Goals

Compute time-frequency representations

- candidate for real-time implementation
- allow adjustments from the user (Levenberg-Marquardt approach)
- allow modes separation and signal reconstruction (synchrosqueezing)
- filter bank approach

Principle

- special case of the STFT using a causal, infinite length window function that can be rewritten as a causal IIR recursive filtering
- the algorithmic complexity depends on the filter order and on the analyzed frequency bandwidth

State-of-the-art

Computing the STFT in terms of recursions

- Recursive STFT [Chen et al.1993], [Amin and Feng, 1995], [S. Tomazic and S.Znidar,1996, 1997]
- STFT and recursive reassignment (implemented using approximative finite differences) [Richard and Lengellé, 1997]
- **STFT and recursive reassignment using a causal infinite window** [**Nilsen**, 2009]

Reassignment

- [Kodera, 1976], [Auger, Flandrin, 1995]
- Levenberg-Marquardt reassigned spectrogram [Auger, Chassande-Mottin and Flandrin2012]

Synchrosqueezing

- synchrosqueezed CWT [Daubechies and Maes, 1996]
- synchrosqueezed STFT [Thakur and Wu, 2011], [Auger et al., 2013]

The STFT as a convolution product

The STFT of a signal x using a window h, denoted $F_x^h(t,\omega) = M_x^h(t,\omega)\,\mathbf{e}^{j\Phi_x^h(t,\omega)}$ can be related to the linear convolution product between the analyzed signal x and the complex valued impulse response of a bandpass filter centered on ω, $g(t,\omega) = h(t)\,\mathbf{e}^{j\omega t}$, $h(t)$ being a real-valued analysis window:

$$y_x^g(t,\omega) = \int_{-\infty}^{+\infty} g(\tau,\omega)x(t-\tau)\,d\tau = |y_x^g(t,\omega)|\,\mathbf{e}^{j\Psi_x^g(t,\omega)} \quad (1)$$

$$= F_x^h(t,\omega)\,\mathbf{e}^{j\omega t} = M_x^h(t,\omega)\,\mathbf{e}^{j(\Phi_x^h(t,\omega)+\omega t)} \quad (2)$$

The STFT is related to the convolution product as

$$M_x^h(t,\omega) = |y_x^g(t,\omega)| \quad (3)$$

$$\Phi_x^h(t,\omega) = \Psi_x^g(t,\omega) - \omega t \quad (4)$$

Rewording the reassignment operators of the spectrogram

According to their definition [Kodera, 1976], [Auger, Flandrin, 1995], the spectrogram reassignment operators can be related to the phase of the STFT and can be reformulated using the phase of $y_x^g(t,\omega)$, denoted $\Psi_x^g(t,\omega) = \Phi_x^h(t,\omega) + \omega t$

$$\hat{t}(t,\omega) = -\frac{\partial \Phi_x^h}{\partial \omega}(t,\omega) = t - \frac{\partial \Psi_x^g}{\partial \omega}(t,\omega), \quad (5)$$

$$\hat{\omega}(t,\omega) = \omega + \frac{\partial \Phi_x^h}{\partial t}(t,\omega) = \frac{\partial \Psi_x^g}{\partial t}(t,\omega). \quad (6)$$

Reassigned spectrogram

$$\mathrm{RSP}(t,\omega) = \iint_{\mathbb{R}^2} |y_x^g(t',\omega')|^2 \delta(t - \hat{t}(t',\omega'))\delta(\omega - \hat{\omega}(t',\omega'))\, dt'\, d\omega' \quad (7)$$

where $\delta(t)$ denotes the Dirac distribution.

Rewording the Levenberg-Marquardt reassignment

[Auger, Chassande-Mottin and Flandrin, 2012]

Levenberg-Marquardt reassignment operators

$$\begin{pmatrix} \tilde{t}(t,\omega) \\ \tilde{\omega}(t,\omega) \end{pmatrix} = \begin{pmatrix} t \\ \omega \end{pmatrix} - \left(\nabla^t R_x^h(t,\omega) + \mu I_2 \right)^{-1} R_x^h(t,\omega) \quad (8)$$

with $R_x^h(t,\omega) = \begin{pmatrix} t - \hat{t}(t,\omega) \\ \omega - \hat{\omega}(t,\omega) \end{pmatrix} = \begin{pmatrix} \frac{\partial \Psi_x^g}{\partial \omega}(t,\omega) \\ \omega - \frac{\partial \Psi_x^g}{\partial t}(t,\omega) \end{pmatrix}$

$$\nabla^t R_x^h(t,\omega) = \begin{pmatrix} \frac{\partial R_x^h}{\partial t}(t,\omega) & \frac{\partial R_x^h}{\partial \omega}(t,\omega) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}(t,\omega) & \frac{\partial^2 \Psi_x^g}{\partial \omega^2}(t,\omega) \\ -\frac{\partial^2 \Psi_x^g}{\partial t^2}(t,\omega) & 1 - \frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}(t,\omega) \end{pmatrix}$$

where I_2 is the 2×2 identity matrix.

Levenberg-Marquardt reassigned spectrogram

$$\text{LMRSP}(t,\omega) = \iint_{\mathbb{R}^2} |y_x^g(t',\omega')|^2 \delta(t - \tilde{t}(t',\omega'))\delta(\omega - \tilde{\omega}(t',\omega'))\, dt'\, d\omega' \quad (9)$$

Rewording the partial derivatives of the phase

$$\frac{\partial \Psi_x^g}{\partial t}(t,\omega) = \mathrm{Im}\left(\frac{y_x^{\mathcal{D}g}(t,\omega)}{y_x^g(t,\omega)}\right) \quad (10)$$

$$\frac{\partial \Psi_x^g}{\partial \omega}(t,\omega) = \mathrm{Re}\left(\frac{y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)}\right) \quad (11)$$

$$\frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}(t,\omega) = \mathrm{Re}\left(\frac{y_x^{\mathcal{D}\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)} - \frac{y_x^{\mathcal{D}g}(t,\omega) y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)^2}\right) \quad (12)$$

$$\frac{\partial^2 \Psi_x^g}{\partial t^2}(t,\omega) = \mathrm{Im}\left(\frac{y_x^{\mathcal{D}^2g}(t,\omega)}{y_x^g(t,\omega)} - \left(\frac{y_x^{\mathcal{D}g}(t,\omega)}{y_x^g(t,\omega)}\right)^2\right) \quad (13)$$

$$\frac{\partial^2 \Psi_x^g}{\partial \omega^2}(t,\omega) = -\mathrm{Im}\left(\frac{y_x^{\mathcal{T}^2g}(t,\omega)}{y_x^g(t,\omega)} - \left(\frac{y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)}\right)^2\right) \quad (14)$$

where y_x^g, $y_x^{\mathcal{T}g}$, $y_x^{\mathcal{D}g}$, $y_x^{\mathcal{D}\mathcal{T}g}$, $y_x^{\mathcal{T}^2g}$ and $y_x^{\mathcal{D}^2g}$ are the outputs of the filters using respectively the impulse responses $g(t,\omega)$, $\mathcal{T}g{=}t\,g(t,\omega)$, $\mathcal{D}g(t,\omega){=}\frac{\partial g}{\partial t}(t,\omega)$, $\mathcal{D}\mathcal{T}g(t,\omega){=}\frac{\partial}{\partial t}\,(t\,g(t,\omega))$, $\mathcal{T}^2g(t,\omega){=}t^2g(t,\omega)$ and $\mathcal{D}^2g(t,\omega){=}\frac{\partial^2 g}{\partial t^2}(t,\omega)$.

Rewording the synchrosqueezed STFT

As previously defined, y_x^g admits the following signal reconstruction formula

$$x(t-t_0) = \frac{1}{h(t_0)} \int_{-\infty}^{+\infty} y_x^g(t,\omega)\, \mathbf{e}^{-j\omega t_0} \frac{d\omega}{2\pi}, \quad \text{when } h(t_0) \neq 0. \tag{15}$$

Synchrosqueezed STFT [Thakur and Wu, 2011], [Auger et al., 2013]

$$\mathrm{Sy}_x^g(t,\omega) = \int_{\mathbb{R}} y_x^g(t,\omega')\, \mathbf{e}^{-j\omega' t_0} \delta\left(\omega - \hat{\omega}(t,\omega')\right)\, d\omega' \tag{16}$$

$\mathrm{LMSy}_x^g(t,\omega)$ is obtained by replacing $\hat{\omega}$ by $\tilde{\omega}$.

Time-frequency representation

A sharpen time-frequency representation is provided by $|\mathrm{Sy}_x^g(t,\omega)|^2$

Signal reconstruction from $\mathrm{Sy}_x^g(t,\omega)$

$$\hat{x}(t-t_0) = \frac{1}{h(t_0)} \int_{\mathbb{R}} \mathrm{Sy}_x^g(t,\omega) \frac{d\omega}{2\pi} \tag{17}$$

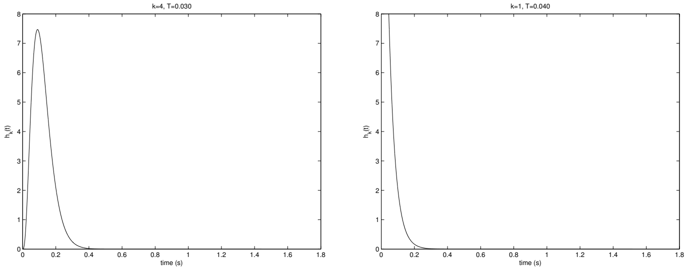
The STFT special case [Nilsen, 2009]

y_x^g can be recursively implemented using

$$h_k(t) = \frac{t^{k-1}}{T^k(k-1)!}\,\mathrm{e}^{-t/T}\,U(t), \quad (18)$$

$$g_k(t,\omega) = h_k(t)\,\mathrm{e}^{j\omega t} = \frac{t^{k-1}}{T^k(k-1)!}\,\mathrm{e}^{pt}\,U(t) \quad (19)$$

with $p = -\frac{1}{T} + j\omega$, $k \geq 1$ being the filter order, T the time spread of the window and $U(t)$ the Heaviside step function.



Discretization using the impulse invariance method [Jackson,2000]

$$G_k(z,\omega) = T_s \mathcal{Z}\left\{g_k(t,\omega)\right\} = \frac{\displaystyle\sum_{i=0}^{k-1} b_i z^{-i}}{\displaystyle 1 + \sum_{i=1}^{k} a_i z^{-i}} \tag{20}$$

with $b_i = \frac{1}{L^{\mathbf{k}}(k-1)!} B_{k-1,k-i-1} \alpha^i$, $\alpha = \mathbf{e}^{pT_s}$, $L = T/T_s$, $a_i = A_{k,i}\,(-\alpha)^i$, T_s being the sampling period. $B_{k,i} = \sum_{j=0}^{i} (-1)^j A_{k+1,j} (i+1-j)^k$ denotes the Eulerian numbers and $A_{k,i} = \binom{k}{i} = \frac{k!}{i!(k-i)!}$ the binomial coefficients.

Hence, using $y_k[n,m] \approx y_x^{g_k}(nT_s, \frac{2\pi m}{MT_s})$ with $n \in \mathbb{Z}$ and $m = 0, 1, ..., M-1$, we obtain

$$y_k[n,m] = \sum_{i=0}^{k-1} b_i\, x[n-i] - \sum_{i=1}^{k} a_i\, y_k[n-i,m] \tag{21}$$

other specific impulse responses

Recursive relations between filters

$$T_s \mathcal{Z}\{\mathcal{T} g_k(t,\omega)\} = kTG_{k+1}(z,\omega) \tag{22}$$

$$T_s \mathcal{Z}\left\{\mathcal{D} g_k(t,\omega)\right\} = \frac{1}{T} G_{k-1}(z,\omega) + pG_k(z,\omega) \tag{23}$$

$$T_s \mathcal{Z}\left\{\mathcal{D}\mathcal{T} g_k(t,\omega)\right\} = k\left(G_k(z,\omega) + pTG_{k+1}(z,\omega)\right) \tag{24}$$

$$T_s \mathcal{Z}\left\{\mathcal{T}^2 g_k(t,\omega)\right\} = k(k+1)T^2 G_{k+2}(z,\omega) \tag{25}$$

$$T_s \mathcal{Z}\left\{\mathcal{D}^2 g_k(t,\omega)\right\} = \frac{1}{T^2} G_{k-2}(z,\omega) + \frac{2p}{T} G_{k-1}(z,\omega) + p^2 G_k(z,\omega) \tag{26}$$

for any $k \geq 1$, provided that $G_0(z,\omega) = G_{-1}(z,\omega) = 0$.

Discrete-time recursive classical reassigned spectrogram

Reassignment operators

$$\hat{n}[n,m] = n - \text{Round}\left(\text{Re}\left(\frac{T_s^{-1} y_x^{\mathcal{T}g}[n,m]}{y_x^g[n,m]}\right)\right) \quad (27)$$

$$\hat{m}[n,m] = \text{Round}\left(\frac{M}{2\pi}\text{Im}\left(\frac{T_s y_x^{\mathcal{D}g}[n,m]}{y_x^g[n,m]}\right)\right) \quad (28)$$

Reassigned spectrogram

$$\text{RSP}_k[n,m] = \sum_{n'\in\mathbb{Z}} \sum_{m'=0}^{M-1} \left|y_k[n',m']\right|^2 \delta\left[n - \hat{n}[n',m']\right] \delta\left[m - \hat{m}[n',m']\right] \quad (29)$$

where $\delta[n]$ is the Kronecker delta

Discrete-time recursive Levenberg-Marquardt reassigned spectrogram

Levenberg-Marquardt reassignment operators

$$\tilde{n}[n,m] = n - \text{Round}\left(\frac{1}{\Lambda}\left(T_s^{-1}\frac{\partial \Psi_x^g}{\partial \omega}\right)\left(1+\mu-\frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}\right) - \frac{1}{\Lambda}\left(T_s^{-2}\frac{\partial^2 \Psi_x^g}{\partial \omega^2}\right)\left(\frac{2\pi m}{M} - T_s\frac{\partial \Psi_x^g}{\partial t}\right)\right) \quad (30)$$

$$\tilde{m}[n,m] = m - \text{Round}\left(\frac{M}{2\pi\Lambda}\left(T_s^{-1}\frac{\partial \Psi_x^g}{\partial \omega}\right)\left(T_s^{2}\frac{\partial^2 \Psi_x^g}{\partial t^2}\right) + \frac{1}{\Lambda}\left(\mu+\frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}\right)\left(m - \frac{M T_s}{2\pi}\frac{\partial \Psi_x^g}{\partial t}\right)\right) \quad (31)$$

with $\Lambda = \left(\mu+\frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}\right)\left(\mu+1-\frac{\partial^2 \Psi_x^g}{\partial t \partial \omega}\right) + \left(T_s^2\frac{\partial^2 \Psi_x^g}{\partial t^2}\right)\left(T_s^{-2}\frac{\partial^2 \Psi_x^g}{\partial \omega^2}\right)$

Hence, $\text{LMRSP}_k[n,m]$ is obtained by replacing $(\hat{n},\hat{m})$ by $(\tilde{n},\tilde{m})$.

Discrete-time recursive synchrosqueezed STFT

Recursive synchrosqueezed STFT

$$\mathrm{Sy}_{k[n,m]} = \sum_{m'=0}^{M-1} y_{k[n,m']} \, \mathbf{e}^{-\frac{2j\pi m' n_0}{M}} \delta\left[m - \hat{m}_{[n,m']}\right] \tag{32}$$

where $n_0 = t_0/T_s$ can be chosen as the time instant $t_0 > 0$ when the maximum of h_k is reached (*i.e.* $n_0 = (k-1)L$). $\mathrm{LMSy}_{k[n,m]}$ is obtained by replacing $(\hat{n}, \hat{m})$ by $(\tilde{n}, \tilde{m})$.

Signal reconstruction from $\mathrm{Sy}_{k[n,m]}$

$$\hat{x}[n - n_0] = \frac{1}{MT_s h_k(n_0 T_s)} \sum_{m=0}^{M-1} \mathrm{Sy}_k[n, m]. \tag{33}$$

For modes extraction, for example [Meignen et al., 2012] can be used.

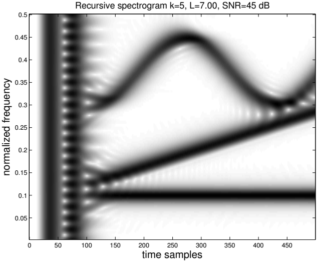
Implementation issue

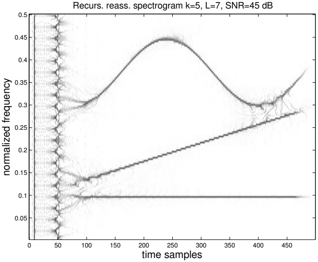
Algorithm for recursive TFR computation

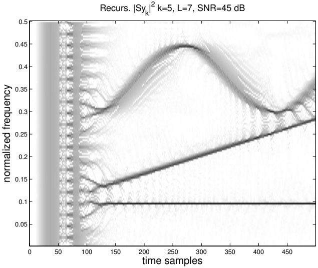
At time index n

1. Compute the required $y_k^g[n,m]$ using $x[n-i]$ and $y_k[n-j,m]$ with $i \in [0, k-1]$, $j \in [1, k]$
2. Compute the other required specific filtered signals (*i.e.* $y_k^{\mathcal{T}g}$, $y_k^{\mathcal{D}g}$, $y_k^{\mathcal{D}\mathcal{T}g}$, $y_k^{\mathcal{T}^2 g}$ or $y_k^{\mathcal{D}^2 g}$) using y_k^g with different filter orders
3. Compute $\hat{n}$,$\hat{m}$ (resp. $\tilde{n}$, $\tilde{m}$) provided by the reassignment operators
4. **If** $\hat{n} \leq n$ (resp. $\tilde{n} \leq n$) **then** update TFR$[\hat{n}, m]$ **otherwise** store the triplet $\left(y_k^g[n,m], \hat{n}, m\right)$ into a list
5. Update TFR$[n, m]$ using all previously stored triplets verifying $\hat{n} \leq n$ and remove them from the list

Time-frequency representations







Signal reconstruction quality

Reconstruction Quality Factor

$$\mathrm{RQF} = 10 \log_{10} \left(\frac{\sum_n |x[n]|^2}{\sum_n |x[n] - \hat{x}[n]|^2} \right) \tag{34}$$

(a)	n_0	8	18	26	**28**	30
	RQF (dB)	9.79	24.17	26.77	**26.82**	26.73
(b)	M	100	200	600	1000	**2400**
	RQF (dB)	20.56	24.90	29.48	30.50	**30.87**
(c)	μ	0.30	0.80	1.30	1.80	**2.30**
	RQF (dB)	20.83	27.28	29.68	30.35	**30.90**

Signal reconstruction quality factor of the recursive synchrosqueezed STFT computed for k = 5, L = 7 at SNR = 45 dB. Line (a), computed for M = 300, Line (b) and Line (c), computed for n0=28 and M=300.

Reference recursive STFT reconstruction RQF = **35.56** dB.

Results

- The first recursive implementation of the synchrosqueezing transform
- A new generalized STFT synthesis formula allowing reconstruction using a causal analysis window
- Adjustable methods allowing to choose a trade-off between the energy concentration and the signal reconstruction quality

Future works

- New practical applications (*e.g.* audio speech enhancement, Music Information Retrieval, Electrical loads recognition, etc.)
- Recursive second-order synchrosqueezing [Oberlin et al., 2015]
- Usage of a different analysis window