On the stochastic modeling of turbulence

the Lagrangian perspective

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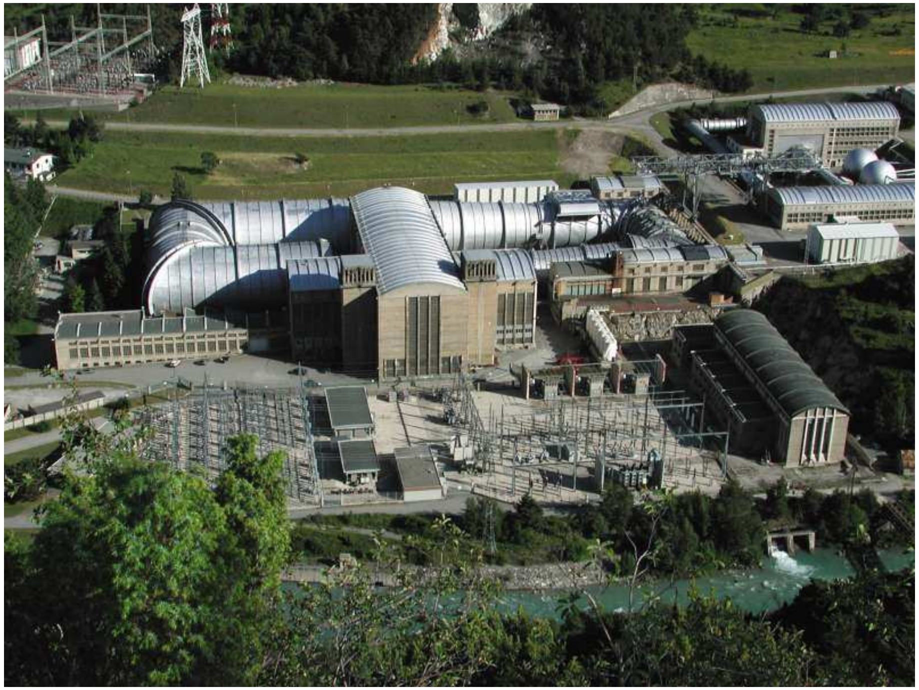
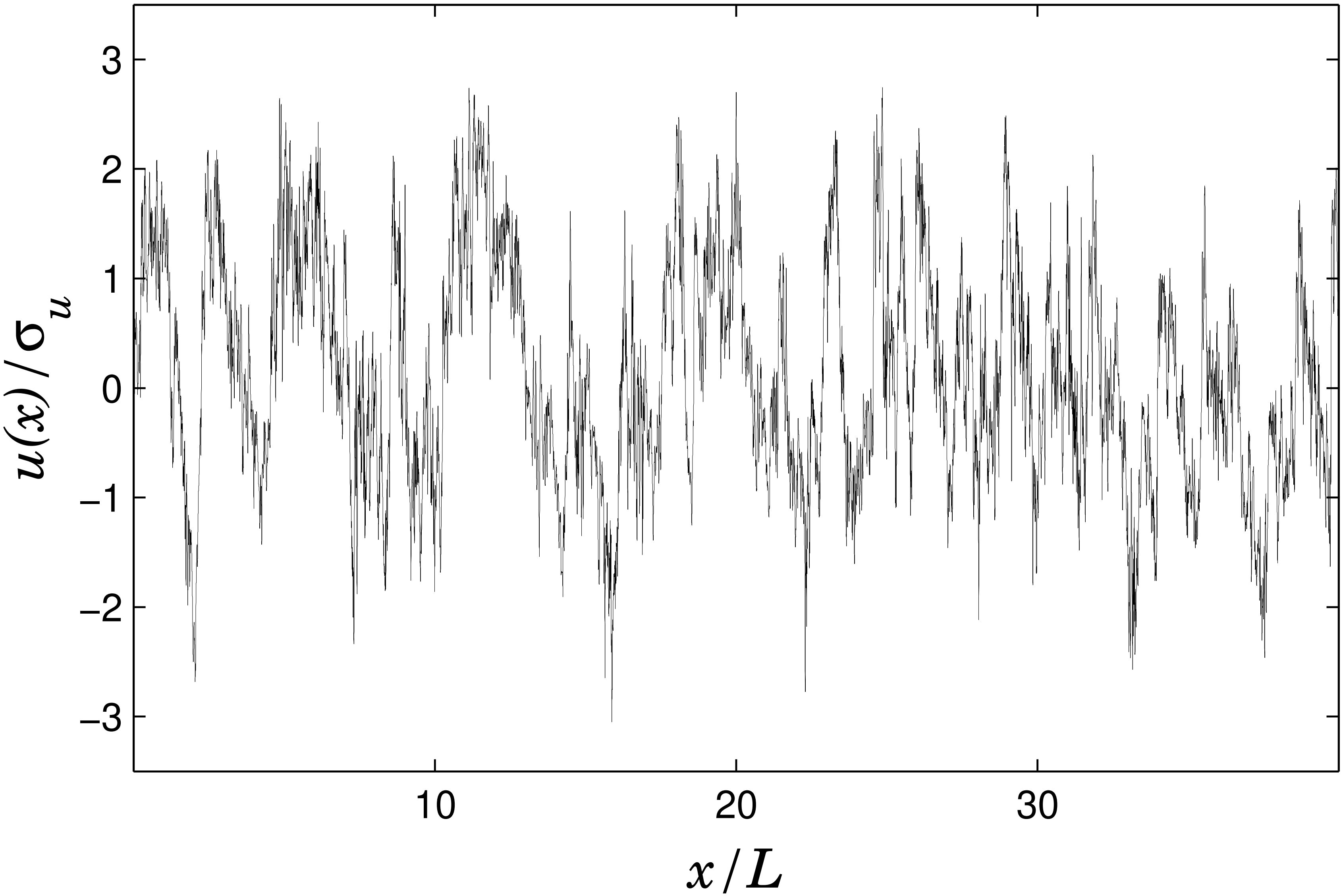
Wind tunnel at Modane



See Gagne et al., Bourgoin et al.

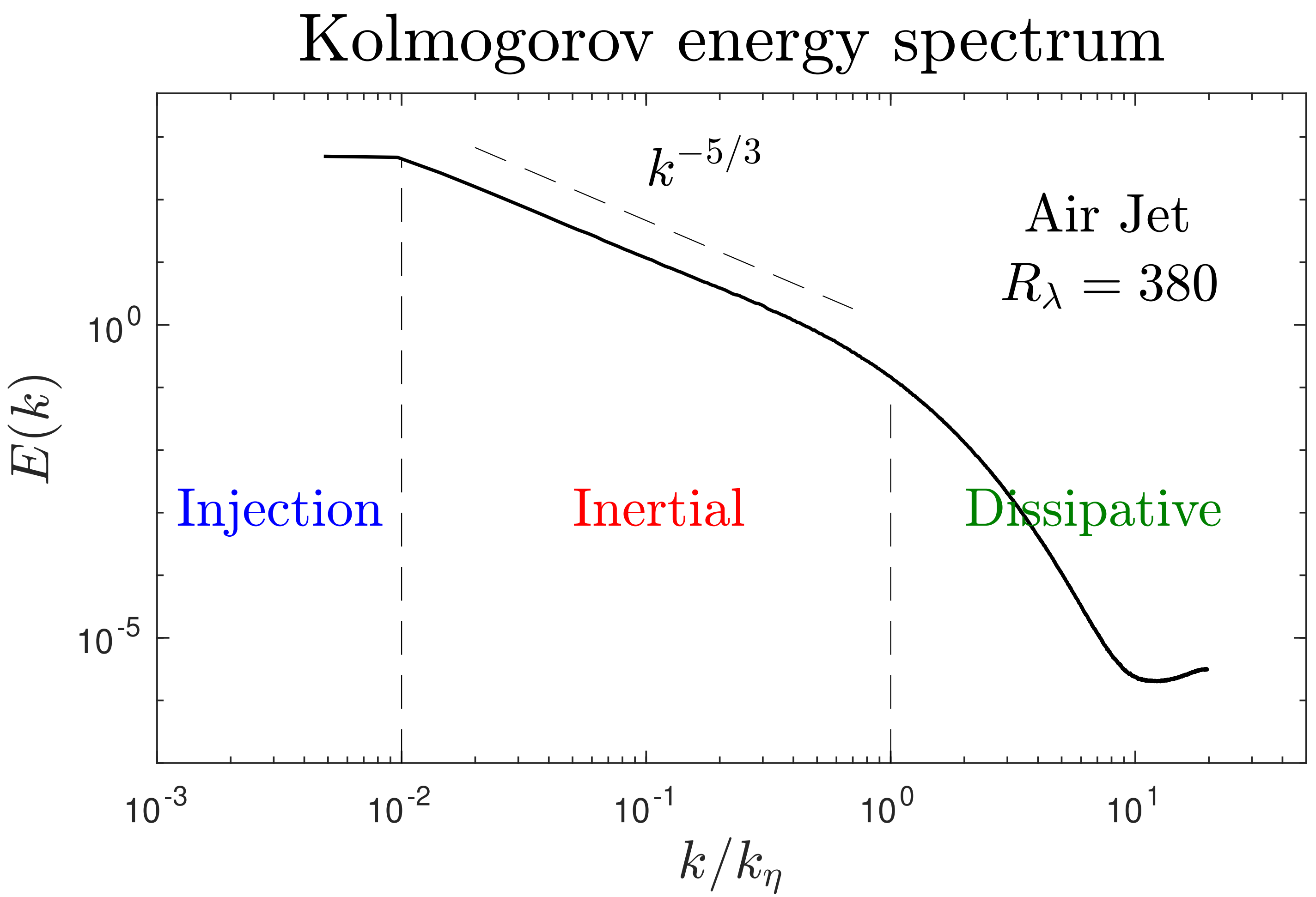
Wind tunnel at Modane



Two-point statistical structure of turbulence

Define the energy spectrum (Fourier transform of the correlation) as

$$E(k) = \int e^{-2i\pi k\ell} \langle u(x)u(x+\ell)\rangle \, d\ell$$

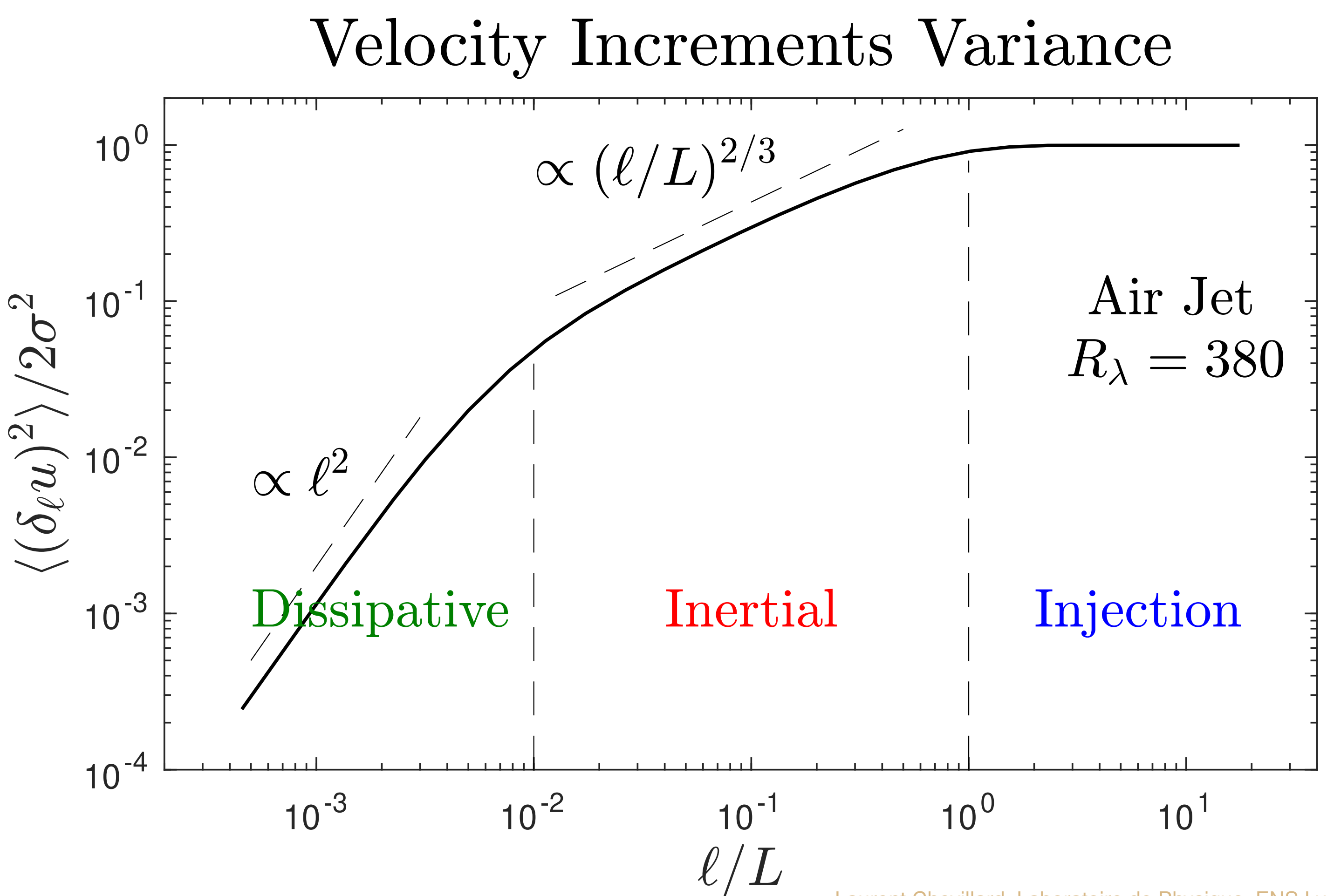


Two-point statistical structure of turbulence

In an equivalent way, define the velocity increment as

$$\delta_\ell u(x) = u(x+\ell) - u(x),$$

and remark that $\langle(\delta_\ell u)^2\rangle = 2\sigma^2 - 2\langle u(x)u(x+\ell)\rangle$.

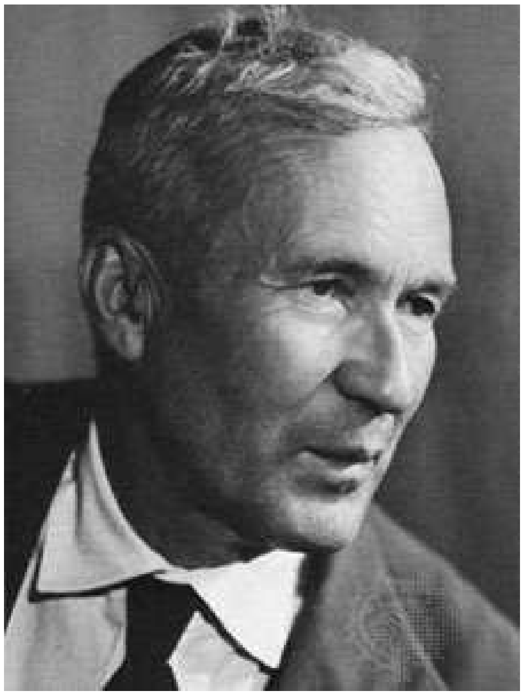


The Navier-Stokes equations

In three-dimensional space, consider the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, where $\boldsymbol{u}=(u_1,u_2,u_3)$, $\boldsymbol{x}\in\mathbb{R}^3$ and say $t>0$. Given a (large-scale, divergence-free forcing) $\boldsymbol{f}$, it is solution of

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}=-\frac{1}{\rho}\boldsymbol{\nabla}p+\nu\Delta\boldsymbol{u}+\boldsymbol{f} \text{ and } \boldsymbol{\nabla}\cdot\boldsymbol{u}=0,$$

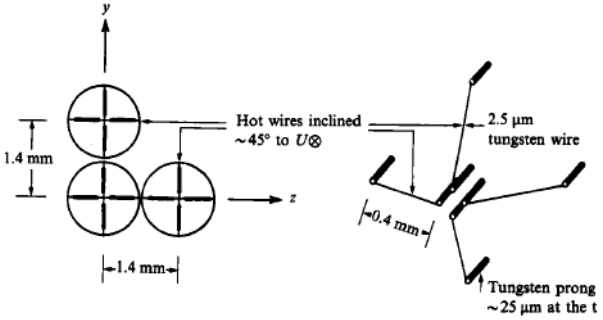
where p is the pressure field, and ν the kinematic viscosity.



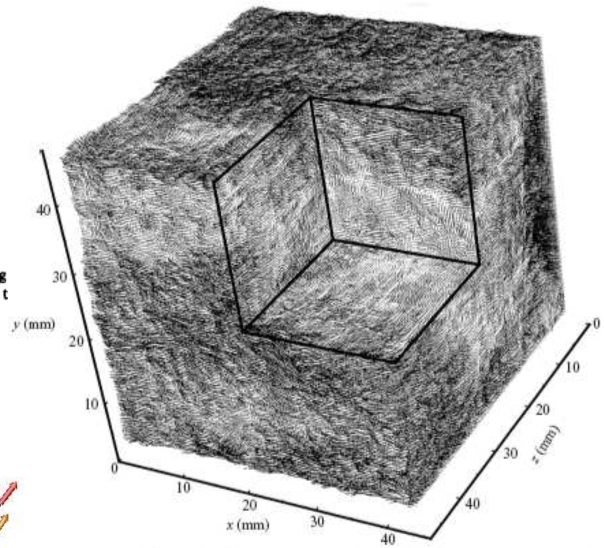
Kolmogorov 1903-1987

"I became interested in turbulent liquid and gas flows at the end of the thirties. From the very beginning it was clear that the theory of random functions of many variables (random fields), whose development only started at that time, must be the underlying mathematical technique. Moreover, I soon understood that there was little hope of developing a pure, closed theory, and because of the absence of such a theory the investigation must be based on hypotheses obtained by processing experimental data."

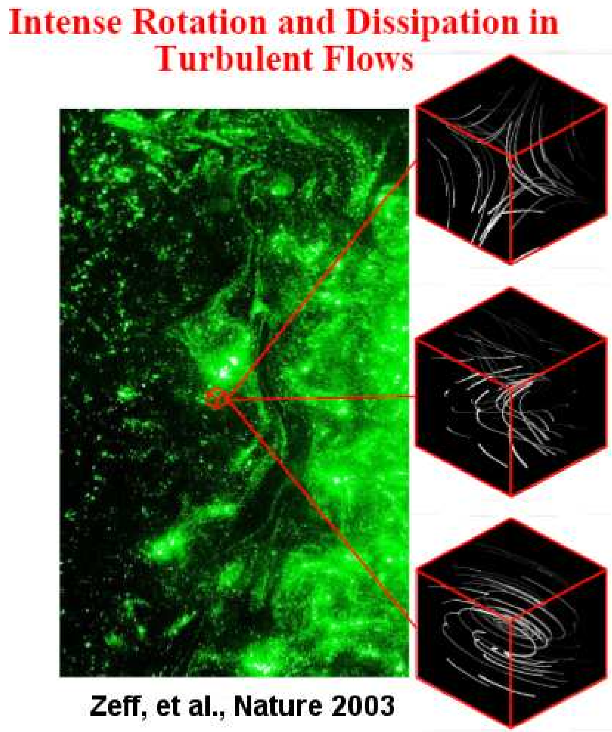
3D *Fluid* ***Turbulence***: *Full* *velocity gradients*



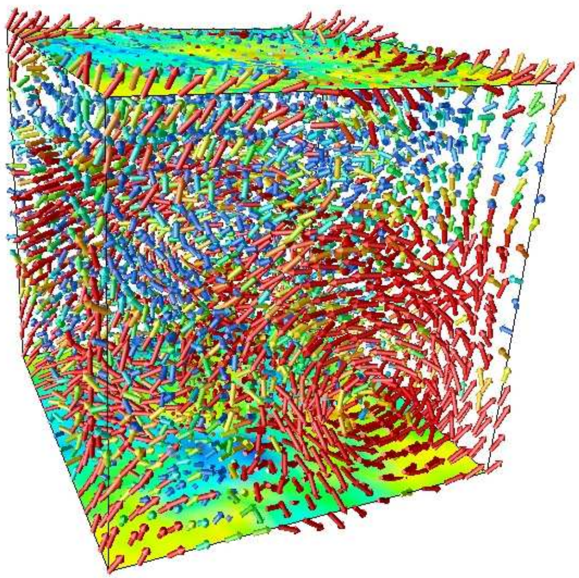
Tsinober, Kit, Dracos J.F.M. 1992



Tao, Katz, Meneveau J.F.M. 2002



Zeff, et al., Nature 2003

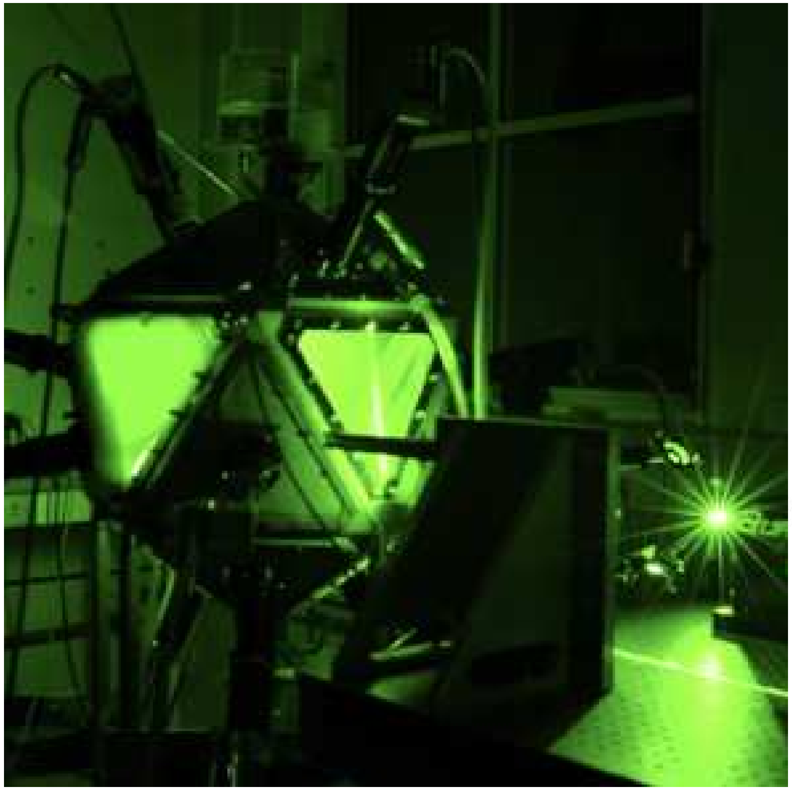


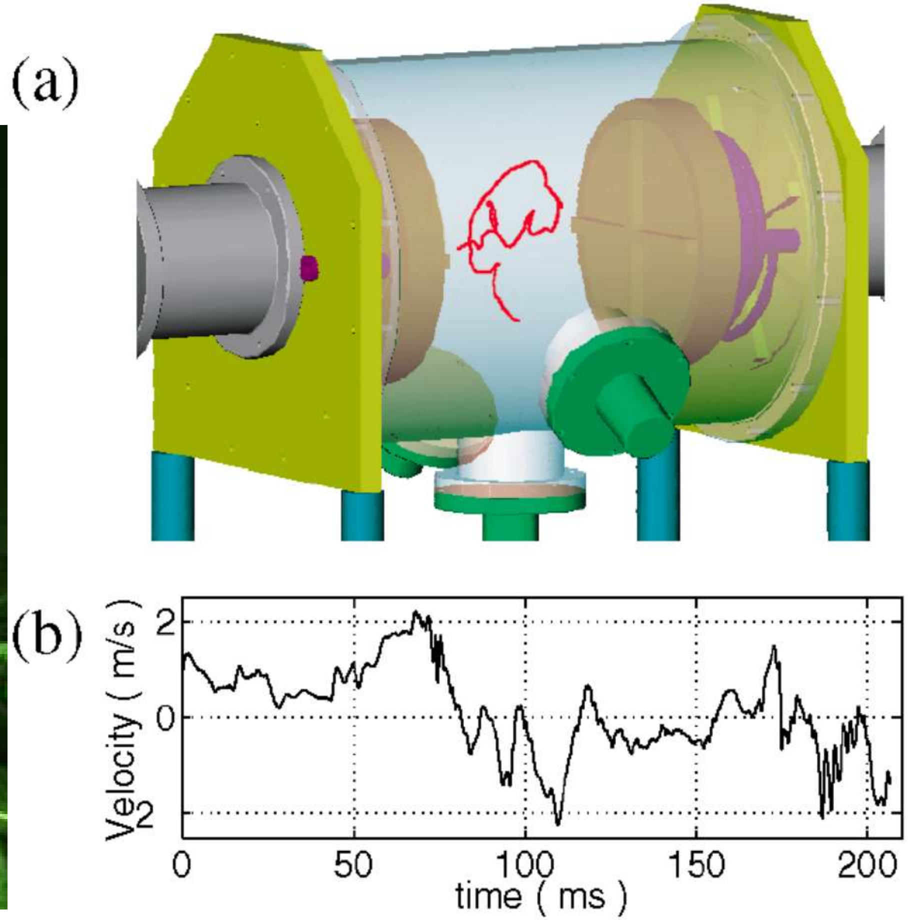
Direct Numerical Simulations
(picture by Toschi)

$$\mathbf{A} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

See also Lüthi et al., Xu-Bodenshatz et al., etc.

The Lagrangian picture



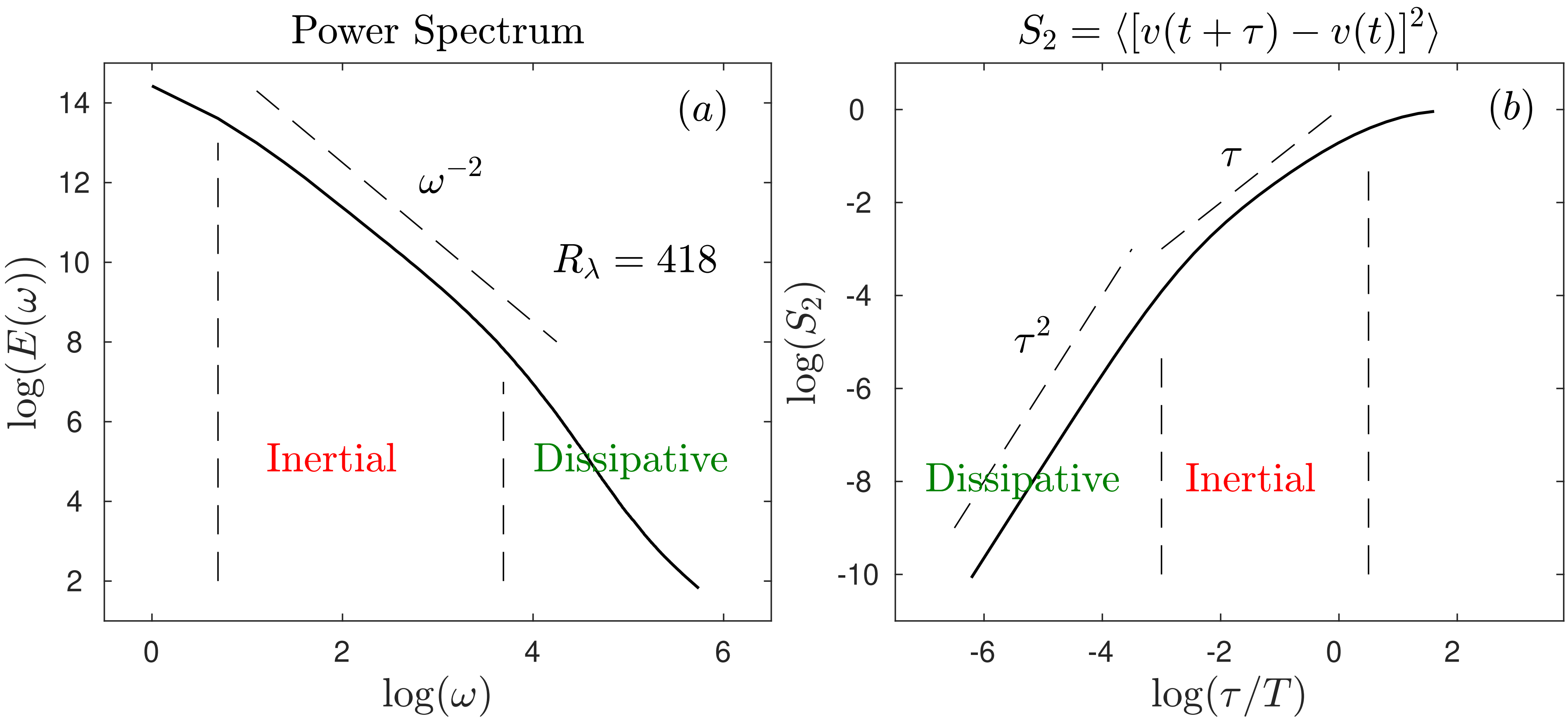


Yeung (97), Mordant et al. (02), Mordant et al. (04), Bourgoin-Volk

Flow equations $v(t) \equiv \dfrac{dX(t)}{dt} = u(X(t), t)$

The Lagrangian picture: Multiscale Analysis

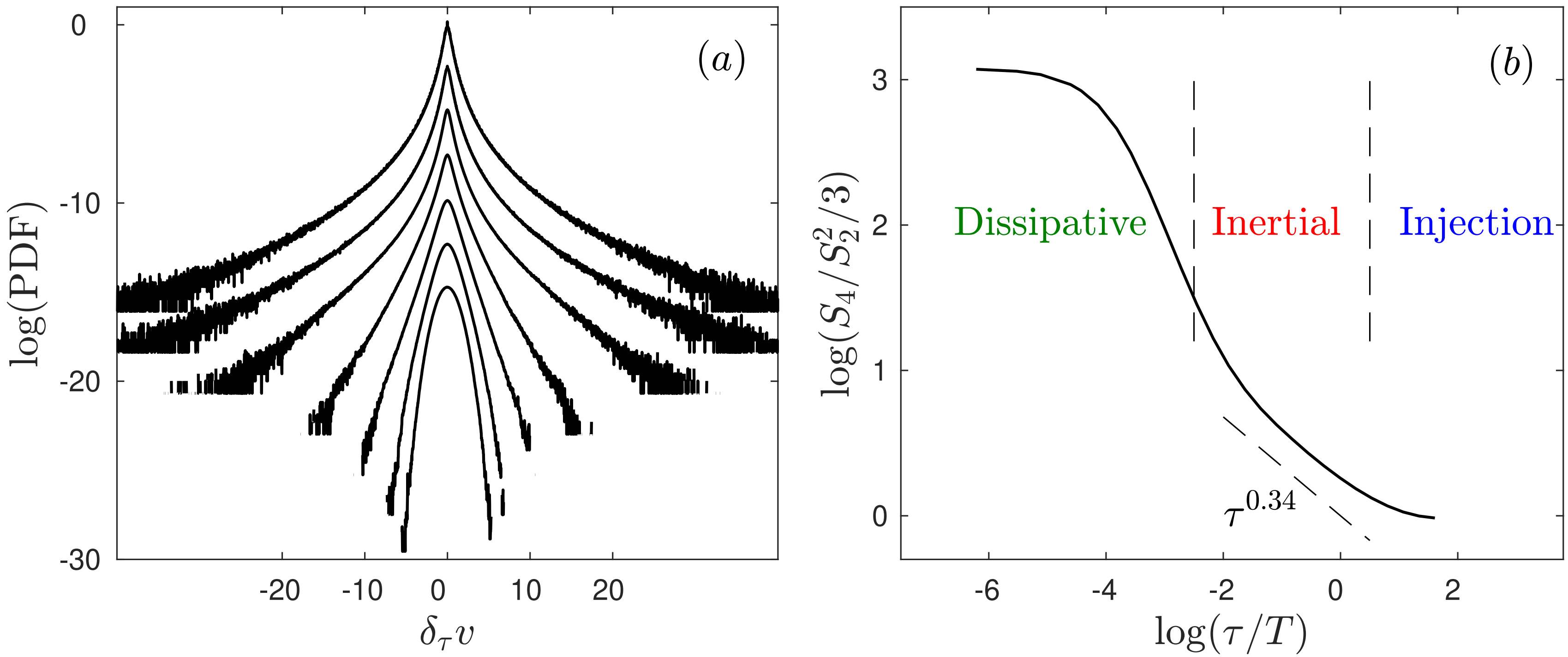
Numerical data from the Hopkins Database: $\mathcal{R}_\lambda = 418$



The Lagrangian picture: Multiscale Analysis

Numerical data from the Hopkins Database: $\mathcal{R}_\lambda = 418$

- The velocity increment: $\delta_\tau v(t) = v(t+\tau) - v(t)$
- We have seen that $\langle (\delta_\tau v)^2 \rangle \propto \tau$ in the inertial range
- We have seen that $\langle (\delta_\tau v)^2 \rangle \approx \tau^2 \langle a^2 \rangle$ in the dissipative range
- What about high-order statistics? such as Probability density functions (PDF) and Flatness?



Asymptotics of phenomenology of fluid turbulence (i)

Consider (as observed) a homogeneous, isotropic **stationary** solution of the (forced over L) Navier and Stokes equations: call it $\mathbf{u}_\nu(x,t)$, with $x \in \mathbb{R}^3$.

- Velocity variance σ^2 is **finite** and **independent** on viscosity ν, i.e.

$$\overbrace{\lim_{\nu\to 0} \mathbb{E}(|\mathbf{u}_\nu|^2)}^{\textbf{Eulerian}} = \overbrace{\lim_{\nu\to 0} \mathbb{E}(|\mathbf{v}_\nu|^2)}^{\textbf{Lagrangian}} = \sigma^2 < +\infty$$

- Consider the **time evolution** of the velocity field $\mathbf{v}_\nu(t)$ along a trajectory (**Lagrangian** description).
  To ensure a bounded velocity variance, the flow will develop small scales:

$$\lim_{\nu\to 0} \mathbb{E}\left[|\mathbf{v}_\nu(t+\tau) - \mathbf{v}_\nu(t)|^2\right] \underset{\tau\to 0}{\propto} \tau,$$

  corresponding to $H = 1/2$ Hölder continuity.

- Similarly, consider the **time evolution** of the velocity field $\mathbf{u}_\nu(x_0,t)$ at a fixed position x_0.

$$\lim_{\nu\to 0} \mathbb{E}\left[|\mathbf{u}_\nu(x_0,t+\tau) - \mathbf{u}_\nu(x_0,t)|^2\right] \underset{\tau\to 0}{\propto} \tau^{2/3},$$

  corresponding to $H = 1/3$ Hölder continuity.

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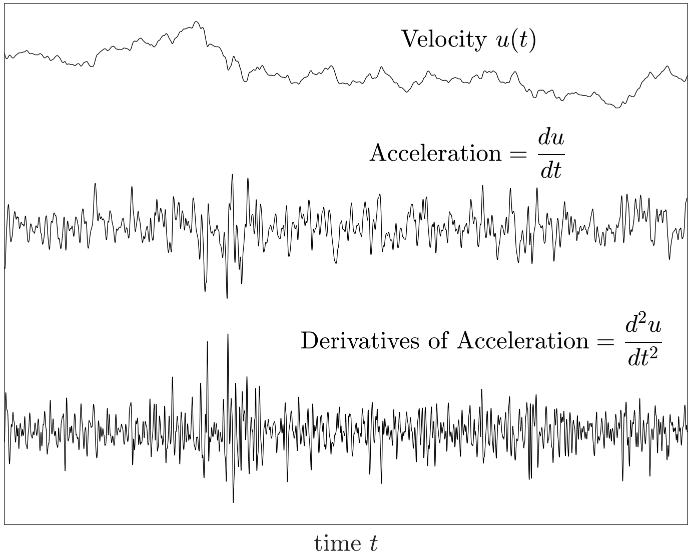
Meaning of these constraints in a (Gaussian) stochastic framework?

- The picture is clear in a **Lagrangian** fashion, since we know well the meaning of the "noise" dW entering in the dynamics of the **Ornstein-Uhlenbeck process**.
- Can we give a meaning to the "noise" $dW_{1/3}$ entering in the dynamics of a *fractional* Ornstein-Uhlenbeck process?, i.e.

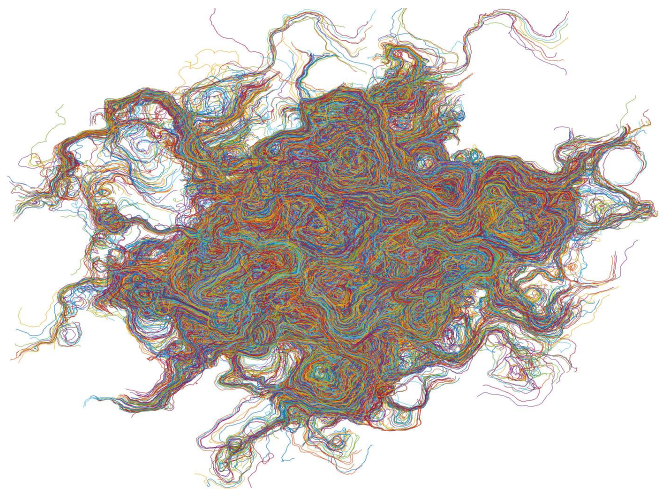
$$du_{1/3}(t) = -\frac{1}{T} u_{1/3}(t)dt + \text{"}dW_{1/3}\text{"}$$

Contributions on the stochastic modeling of Lagrangian turbulence



Can we build up an infinitely differentiable and causal random process to mimic fluctuations of Lagrangian velocity at a finite Reynolds number?
$\rightarrow$ B. Viggiano, J. Friedrich, R. Volk, M. Bourgoin, RB Cal, L. Chevillard (2019).

What are the minimal ingredients to include in a spatio-temporal random advecting Eulerian field such that induced Lagrangian velocities are realistic of experimental observations?
$\rightarrow$ J. Reneuve, L. Chevillard (2020).

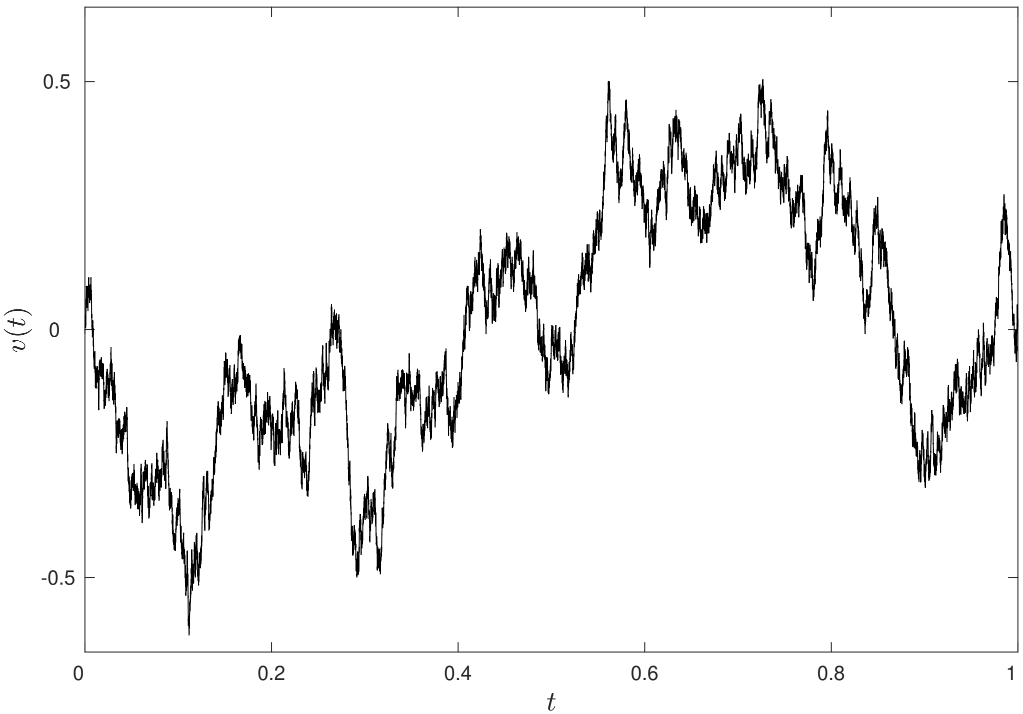


Ornstein-Uhlenbeck processes

- Consider the following linear stochastic differential equation

$$dv_1(t) = -\frac{1}{T}v_1(t)dt + \sqrt{q}W(dt) \equiv a_1(t),$$

- where T is meant to be the large ($\sim$ integral) timescale
- $\rightarrow$ velocity profile $v_1(t)$ not differentiable, but proper asymptotic regularity
- $\rightarrow$ acceleration $a_1(t)$ is a random distribution
- $\rightarrow$ ask for including finite Reynolds number effects



(two-layered) Ornstein-Uhlenbeck processes

$\rightarrow$ (Sawford 91)

- Consider the following linear stochastic differential equation

$$\frac{dv_2}{dt} = -\frac{1}{T}v_2(t) + f_1(t) \equiv a_2(t)$$

$$df_1(t) = -\frac{1}{\tau_\eta}f_1(t)dt + \sqrt{q}W(dt)$$

- where τ_η is meant to be the small ($\sim$ dissipative) timescale
- $\rightarrow$ velocity profile $v_1(t)$ differentiable, but proper asymptotic regularity
- $\rightarrow$ acceleration $a_2(t)$ is a <u>classical</u> random function (finite variance), but not differentiable

Acceleration correlation function in DNS

→ (see Lamorgese, Pope, Yeung, Sawford 2007)

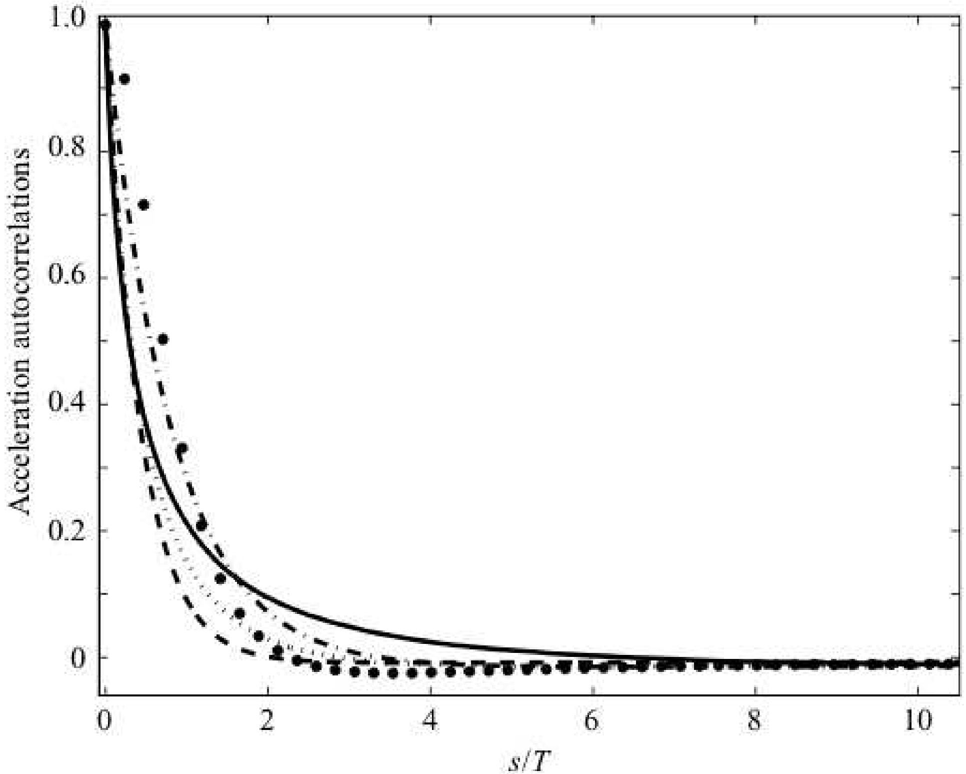


FIGURE 8. Acceleration autocorrelations from CCG simulations based on (5.2)–(5.3) (solid), Sawford 1991 (dashed), Sawford 1991 with a_0 from Sawford *et al.* (2003) (dot-dashed) and Reynolds 2003 (dotted) models compared to component-averaged data at $R_\lambda \approx 650$ from 2048^3 DNS (symbols).

Making velocity infinitely differentiable

$\rightarrow$ B. Viggiano, J. Friedrich, R. Volk, M. Bourgoin, RB Cal, L. Chevillard (2019).
$\rightarrow$ So, in a Gaussian framework, it is tempting to consider the following system of embedded sdes, for $n \rightarrow \infty$,

$$
\begin{aligned}
\frac{dv_n}{dt} &= -\frac{1}{T} v_n(t) + f_{n-1}(t) \equiv a_n(t) \\
\frac{df_{n-1}}{dt} &= -\frac{1}{\tau_\eta} f_{n-1}(t) + f_{n-2}(t) \\
&\dots \\
\frac{df_2}{dt} &= -\frac{1}{\tau_\eta} f_2(t) + f_1(t) \\
df_1 &= -\frac{1}{\tau_\eta} f_1(t)dt + \sqrt{q_{(n)}} W(dt) \ .
\end{aligned}
$$

Making velocity infinitely differentiable (properly)

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$$\frac{dv_n}{dt} = -\frac{1}{T}v_n(t) + f_{n-1}(t) \equiv a_n(t)$$

$$\frac{df_{n-1}}{dt} = -\frac{\sqrt{n-1}}{\tau_\eta}f_{n-1}(t) + f_{n-2}(t)$$

$$\dots$$

$$\frac{df_2}{dt} = -\frac{\sqrt{n-1}}{\tau_\eta}f_2(t) + f_1(t)$$

$$df_1 = -\frac{\sqrt{n-1}}{\tau_\eta}f_1(t)dt + \sqrt{\alpha_n}W(dt) \ ,$$

with

$$\alpha_n = \left(\frac{n-1}{\tau_\eta^2}\right)^{n-1} \frac{2\sigma^2 e^{-\tau_\eta^2/T^2}}{T \text{erfc}\left(\tau_\eta/T\right)}.$$

Making velocity infinitely differentiable (properly)

$$\mathcal{C}_{v_n}(\tau) = \frac{2\sigma^2 e^{-\tau_\eta^2/T^2}}{T\text{erfc}\,(\tau_\eta/T)} \int_{\mathbb{R}} e^{2i\pi\omega\tau} \frac{T^2}{1+4\pi^2T^2\omega^2} \left[\frac{1}{1+\frac{4\pi^2\tau_\eta^2\omega^2}{n-1}} \right]^{n-1} d\omega,$$

such that

$$\mathcal{C}_v(\tau) \equiv \lim_{n\to\infty} \mathcal{C}_{v_n}(\tau) = \frac{2\sigma^2 e^{-\tau_\eta^2/T^2}}{T\text{erfc}\,(\tau_\eta/T)} \int_{\mathbb{R}} e^{2i\pi\omega\tau} \frac{T^2}{1+4\pi^2T^2\omega^2} e^{-4\pi^2\tau_\eta^2\omega^2} d\omega.$$

- then $\mathcal{C}_v(\tau)$ and $\mathcal{C}_a(\tau)$ can be explicitly derived (simple).

Go on (and never stop): Including ***Intermittency***

$\rightarrow$ (generalization of the multifractal random walk of Bacry, Delour, Muzy 2001)

$$\frac{dv_{n,\epsilon}}{dt} = -\frac{1}{T}v_{n,\epsilon}(t) + e^{\gamma X_{n,\epsilon}(t) - \frac{\gamma^2}{2}\langle X^2_{n,\epsilon}\rangle} f_{n-1}(t) \equiv a_n(t)$$

$$\frac{df_{n-1}}{dt} = -\frac{\sqrt{n-1}}{\tau_\eta} f_{n-1}(t) + f_{n-2}(t)$$

$$\dots$$

$$\frac{df_2}{dt} = -\frac{\sqrt{n-1}}{\tau_\eta} f_2(t) + f_1(t)$$

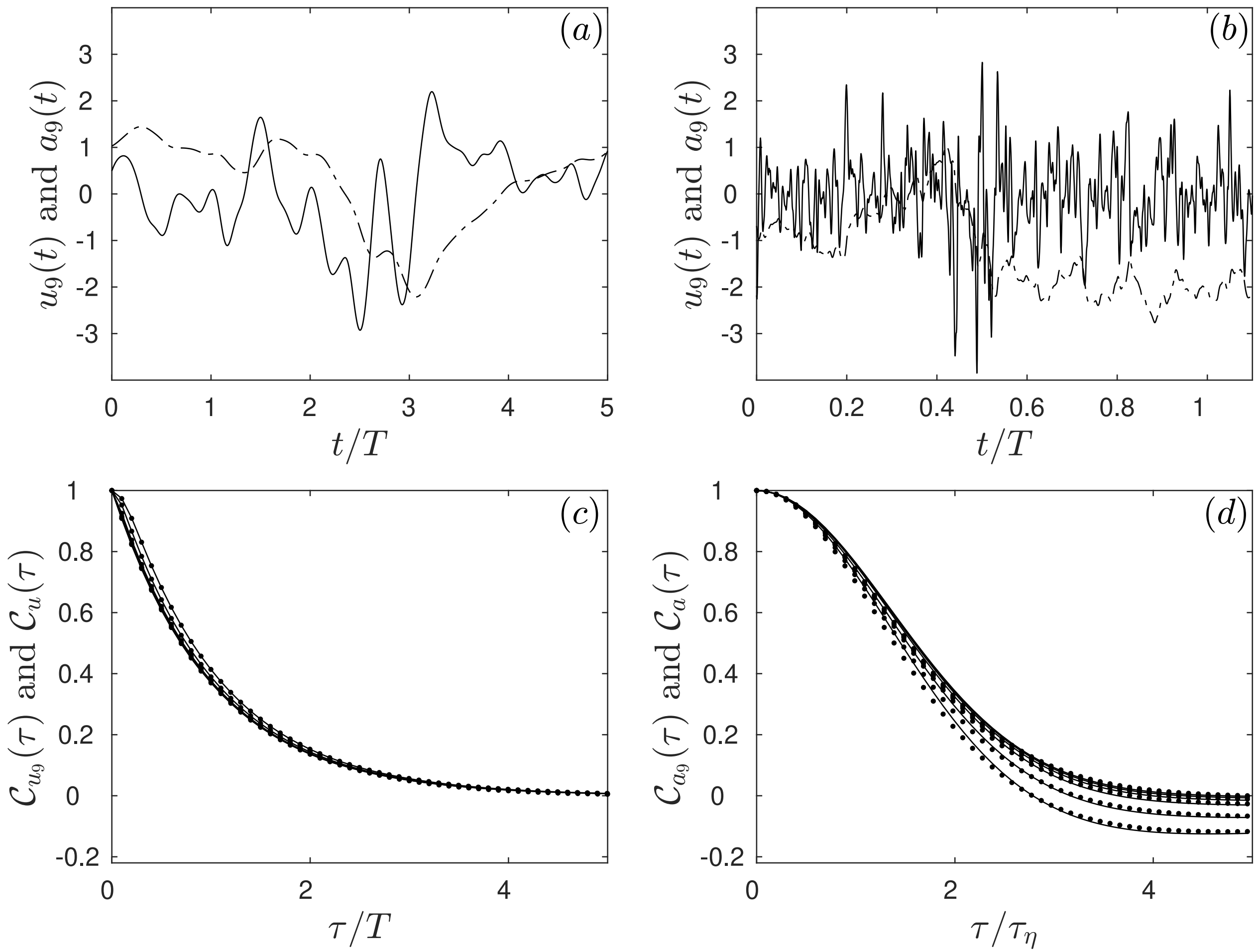
$$df_1 = -\frac{\sqrt{n-1}}{\tau_\eta} f_1(t)dt + \sqrt{\beta_n} W(dt) \ ,$$

with

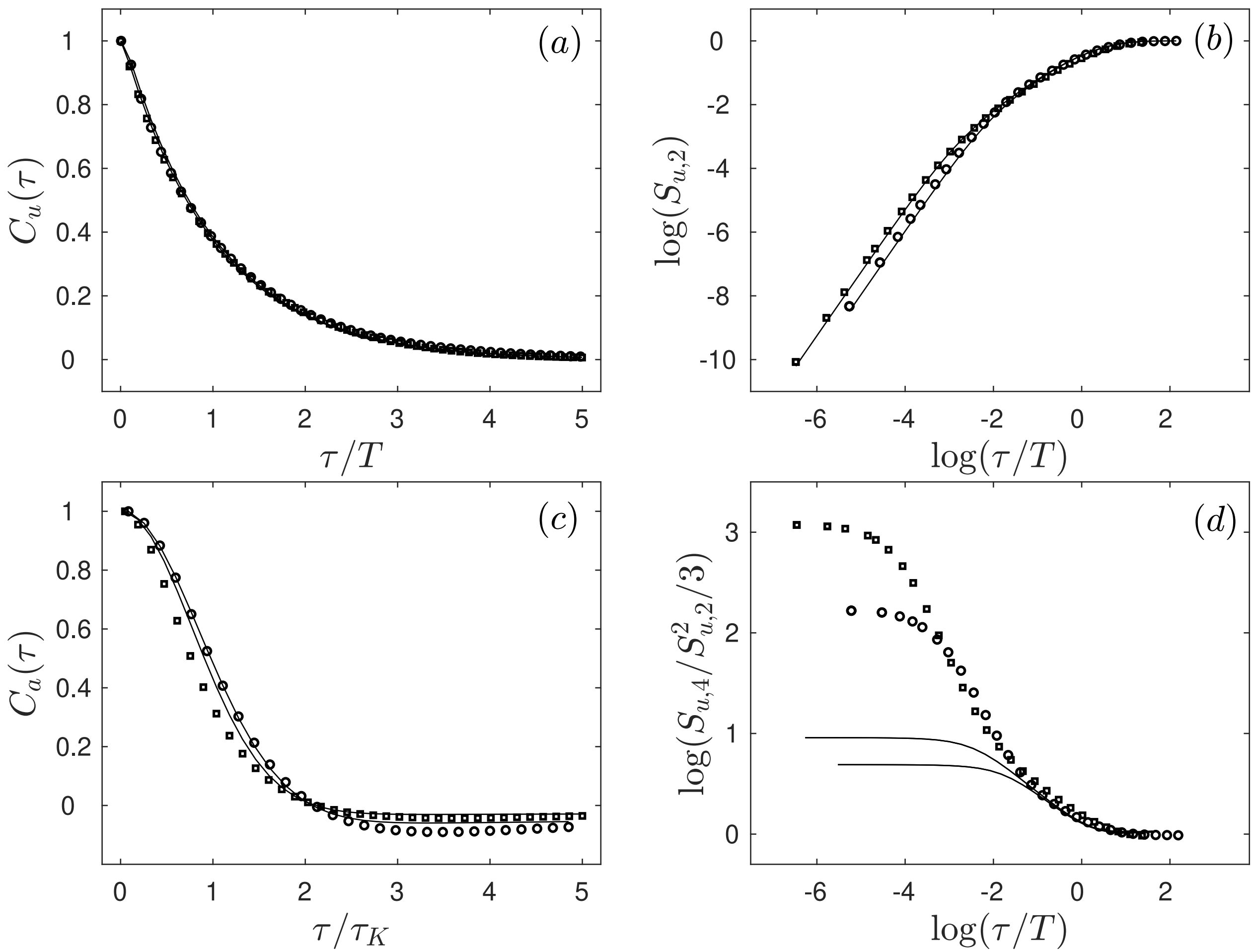
$$\beta_n = \left(\frac{n-1}{\tau_\eta^2}\right)^{n-1} \frac{\sigma^2\sqrt{4\pi\tau_\eta^2}}{T\int_0^\infty e^{-\frac{h}{T}} e^{-h^2/(4\tau_\eta^2)} e^{\gamma^2 \mathcal{C}_X(h)} dh} .$$

- $X_{n,\epsilon}$ is a nth-layered regularized ($\epsilon > 0$) **fractional** Ornstein-Uhlenbeck process of vanishing Hurst exponent
- and its exponential converges towards a multifractal measure (Kahane 87)
- then $\mathcal{C}_v(\tau)$ and $\mathcal{C}_a(\tau)$ can be explicitly derived.

Numerical simulations of the obtained random process



Comparisons to DNS data



Origin	Resolution	$\mathcal{R}_\lambda$	τ_K	T_L	number of trajectories	dt	Duration
Turbase	512^3	185	0.0470	0.7736	126720	4.10^{-3}	$17.063\,T_L$
JHTDB	1024^3	418	0.0424	1.3003	32768	2.10^{-3}	$7.692\,T_L$

TABLE 1. Summary of relevant physical parameters of the two sets of DNS data. Resolution of the Eulerian fields, Taylor based Reynolds number $\mathcal{R}_\lambda$ and Kolmogorov dissipative timescale τ_K (Eq. 4.2) are provided in relevant publications (see text). The Lagrangian integral timescale T_L is defined in Eq. 4.1 and is computed from on our statistical estimation of the velocity correlation function.

The Era of Random Fields

$\rightarrow$ From stochastic processes to random fields
$\rightarrow$ From Causality to statistical Homogeneity and/or Isotropy
$\rightarrow$ From Lagrangian velocity $v(t)$ to a spatio-temporal Eulerian vector field $\mathbf{u}(\mathbf{x}, t)$
$\rightarrow$ Start with the Gaussian view of things, and a spatio-temporal white noise $W(d^d x, dt)$.

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$\rightarrow$ Start with the Gaussian view of things, and a spatio-temporal white noise $W(d^d x, dt)$.

$\rightarrow$ Keep in mind that we will eventually be interested in the **flow equations**

$$\mathbf{v}(t) \equiv \frac{d\mathbf{X}(t)}{dt} = \mathbf{u}\left(\mathbf{X}(t), t\right)$$

considering incompressible (divergence-free) advecting Eulerian fields $\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0$.
$\rightarrow$ For this reason, consider $d = 2$ spatial dimension.

The Era of Random Fields

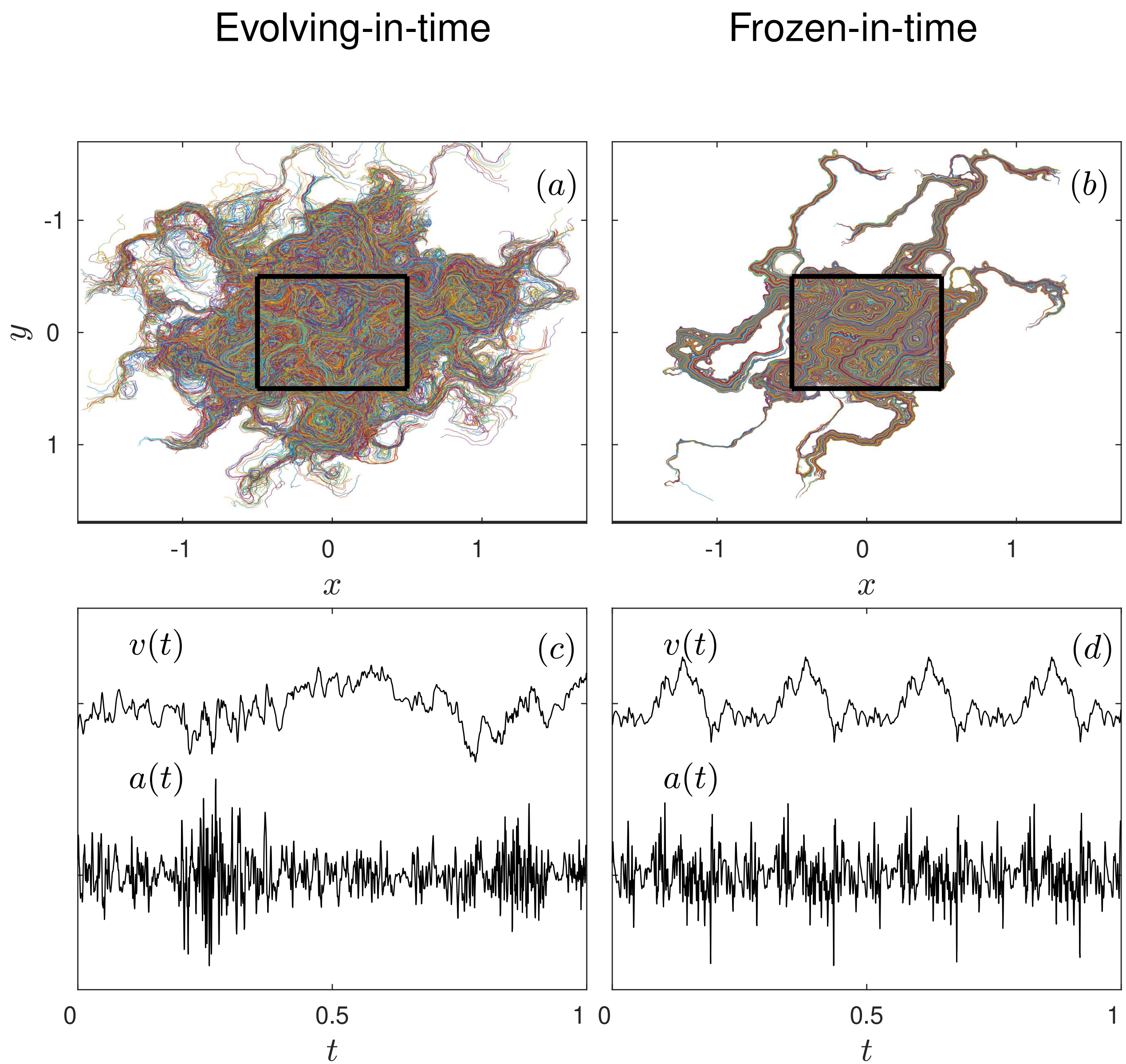
$\rightarrow$ Consider then an incompressible, statistically homogeneous, isotropic and stationary velocity field with proper regularity H in both space and time,

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_{\boldsymbol{y}\in\mathbb{R}^2, s\in\mathbb{R}} \boldsymbol{\mathcal{G}}_H(\boldsymbol{x}-\boldsymbol{y}, t-s) W(d^2y, ds)$$

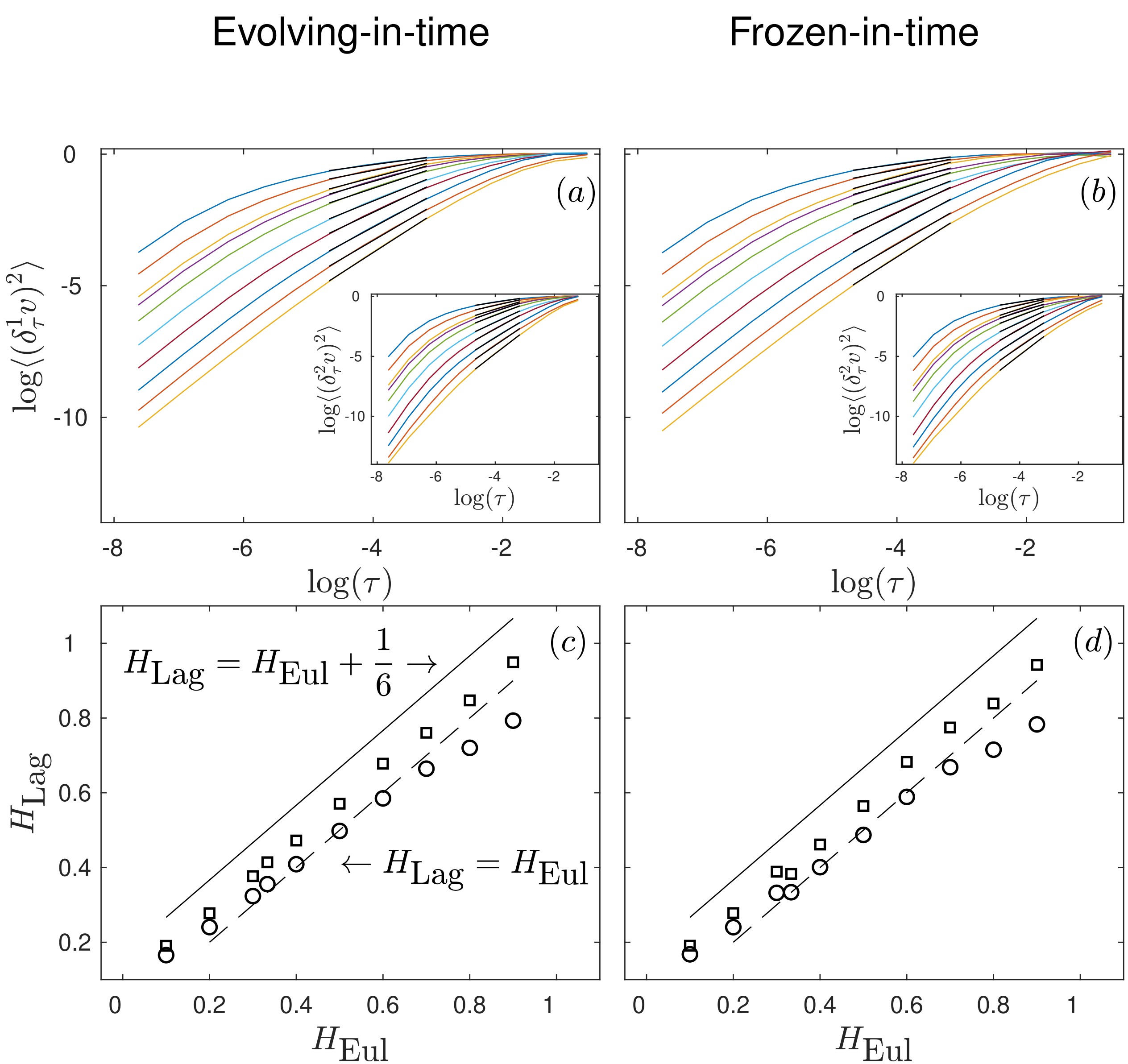
$$\boldsymbol{\mathcal{G}}_H(\boldsymbol{x},t) = \varphi(\boldsymbol{x},t)\frac{\boldsymbol{x}^\perp}{|\boldsymbol{x}|}||\boldsymbol{x},t||^{H-3/2}$$

- A functional form inspired by the Biot-Savart law.
- $||\boldsymbol{x},t||^2 = |\boldsymbol{x}|^2 + \sigma^2 t^2$ a spatio-temporal norm.
- φ a spatio-temporal cut-off function over large (integral) L and T scales.
- H the Holderian regularity, $H \approx 1/3$ for turbulence.
- Keep in mind that this has to be regularized over a small scale ϵ to ensure differentiability.
- then do funky movies.
- See also alternative (Markovian) propositions by Chavez-Gawedzki-Horvai-Kupiainen-Vergassola (2003).

Solving the flow equations



and measure the regularity of Lagrangian velocity



Conclusions

- Whereas the stochastic modeling of Lagrangian velocity can be done with great success (B. Viggiano et al. arXiv:1909.09489 (2019))
- It remains to understand why and how $\frac{1}{3}$-Eulerian regularity makes a $\frac{1}{2}$-Lagrangian regularity.
- Note also intermittent corrections on v while u is Gaussian.
- See J. Reneuve et al. arXiv:2004.02864 (2020)

Intermittency in Eulerian fluctuations

Eulerian longitudinal velocity increments: $\delta_\ell u(x) = u(x+\ell) - u(x)$

Flatness $F = \frac{\langle (\delta_\ell u)^4 \rangle}{\langle (\delta_\ell u)^2 \rangle^2}$