The problem of time in quantum gravity : some perspectives in finite-dimensional models

Sylvain Carrozza

ENS de Lyon,
& Lyon, France

Institute for Gravitation and the Cosmos,
& Physics Department, Penn State,
University Park, PA 16802, U.S.A.

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Introduction

The problem of time in quantum gravity has focused a lot of attention during the last decades. It emerged from early Hamiltonian tentative quantizations of General Relativity (GR), such as Bryce DeWitt's superspace formalism [1]. Later, in the eighties, the introduction of Ashtekar's variables has driven new work in the field, known has Loop Quantum Gravity (LQG)[2–4]. A symmetry reduced cosmological model, known as Loop Quantum Cosmology (LQC), and mainly developed in Penn State by Ashtekar, Bojowald, and collaborators [5, 6], revealed to solve one major drawback of DeWitt's theory : in LQC, the initial Big Bang singularity is replaced by a *Big Bounce*, where physical quantities remain well-defined [7–9]. LQG is now considered, with String theory, as a main path to a consistant theory of quantum gravity.
But despite the large amount of work and results coming from LQG, most of the initial questions about the nature of time remain. Indeed, they would emerge from any Hamiltonian quantization of GR. In such approaches [10], the classical theory is written as a constraint Hamiltonian system. The fundamental equations are a set of constraints, coding gauge invariance and dynamics. The latter is generally called *Hamilton constraint*, and noted H. The equation of motion is then simply $H = 0$. Once quantized, this become : $\widehat{H}\Psi = 0$, where Ψ is a vector in a kinematical Hilbert space $\mathcal{H}$, and $\widehat{H}$ an operator on it. So any state annihilated by $\widehat{H}$ (and by the gauge constraints) is a physical state, representing a whole history of the system. There is no explicit time variable similar to what we have in non-relativistic quantum mechanics. Moreover, any physical observable (that is operators leaving the space of physical states invariant) must commute with H. It's therefore a constant of motion, and we end up with a *frozen formalism.* This is known as the problem of time in quantum gravity [10] : how a frozen formalism can account for change? *Prima facie* we have to face a striking contradiction with an elementary and experimentally assessed notion : change.
In the first two sections, we'll present in details the problem of time at both classical and quantum levels, and discuss several approaches to solve it. All these approaches are in our view very relevant and suffer from severe drawbacks, even if sometimes mainly technical. That's why we choose to briefly discuss all of them. The two other sections will sum up some of the work we did for nearly four months at the Institute for Gravitation and the Cosmos, under Abhay Ashtekar's supervision. More precisely, we'll first present a personal study of time in non-relativistic quantum mechanics. We will then discuss some aspects of time in cosmological models. This last part will present ideas and calculations from a collaborative work with Marc Geiller.

I. REVIEW OF ISSUES AND RESULTS AT THE CLASSICAL LEVEL

In this section, we'll present the general Hamiltonian formalism for gauge systems [3, 10], used in the loop representation of GR. This will allow us to discuss the notion of observable, and show that, even at the classical level, we'll need to use caution if we want to give a consistant picture of time and evolution. This will also open the path to the following section, which will deal with quantum aspects of the problem. In this section, and even in the following, we'll often refer to Carlo Rovelli's work [11–14], which in our view gives a very clear and satisfactory solution at the classical level, and many hints at the quantum level.

A. Presymplectic system

A presymplectic system is a generalization of a symplectic one, where we allow the fundamental two-form to be degenerated. It's the proper formalism to discuss constraint Hamiltonian systems. For simplicity we'll assume $\mathcal{S}$ to be a physical system with a finite number of degrees of freedom, and described by a unique constraint.[1] Let $\mathcal{C}$ be the configuration space, and $\widetilde{\omega}$ the canonical symplectic form on $T^*\mathcal{C}$. As usual, we'll note $\{\ ,\ \}$ the associated Poisson brackets. The constraint H is a function on $T^*\mathcal{C}$, defining the constraint surface Σ, on which $\widetilde{\omega}$ induces the presymplectic form ω. Motions of the system are generated by null vector fields of ω. They form the phase space Γ. If $\mathcal{C}$ is n-dimensional, Σ is $2n-1$-dimensional, and Γ is $2(n-1)$-dimensional.[2]
The main motivation to use presymplectic structures to describe relativistic situations is that we can include the time variable in the configuration space. For instance, configurations of a particle moving on a line are couples (x, t) of its position and time. T^*C is then coordinatized by the variables $\{x, t, p_x, p_t\}$. In the non-relativistic case, the constraint is simply $H = p_t + h$ where h is the usual Hamiltonian describing the dynamics. The two-form ω is shown to be $\omega = dp_x \wedge dx - dh \wedge dt$. It's then easy to check that solving $\omega(X, .) = 0$ on the constraint surface amounts to solve the Hamilton equations $\widetilde{\omega}(X, .) = -dh$, so that we recover the right dynamics of the system.
This procedure is general, and allows to put time and space variables on the same footing from the beginning. This opens the next discussion, on the notion of observable in classical mechanics.

B. *Partial* and *complete* observables

As Rovelli pointed out [3, 15], even at the classical level, we have to be careful with what we call *observable*. The most natural definition is : an observable is a quantity for which we can make predictions, and conceive experimental procedures to measure it. But contrary to what we would be tempted to think, this doesn't mean that time and space, nor any configuration variable of a classical system, are observable. For instance, there is no way of predicting and no empirical procedure for measuring time alone. This would amount to answer the question : what is the value of time t? Of course we need additional specifications. For example : "at what value t of time such or such event occurs?" is a well-formulated question. Indeed, in every experiment, what we really measure are correlations, such as the position of a particle in function of time. We call them *complete observables*, as

[1] LQG has obviously an infinity of degrees of freedom, and is described by three sets of constraints. However, in LQC we are left with a finite number of degrees of freedom, and one constraint. There will be no technical subtleties in this case.
[2] In the case of N constraints, we end up with a $2(n-N)$-dimensional phase space.

opposed to coordinates on $\mathcal{C}$ which we'll call *partial observables*. There is no way of predicting and measuring positions or time alone, but we can compute and observe correlation between them. It's worth-noting at this point that, in particular, all of our experience of time is based on such correlations : position of the sun in the sky, oscillations of a pendulum, vibrations of a crystal etc.
With respect to the symplectic structure of $T^*\mathcal{C}$, partial and complete observables are really different. The latter commute with the constraint, and are therefore constant of motions in the sense of Hamiltonian physics. Indeed, if we fix a specific value t of time,the complete observable $x(t)$ is constant on any trajectory of the system. For the general case in which the constraint is not of the form $p_t + h$, complete observables are functions on $T^*\mathcal{C}$ which commute with the constraint.[3]

C. Flow of time and complete observables

We are now in position to address the problem of time at the classical level. Let $\{\mathcal{C}\,,\,H\}$ be a dynamical system. Complete observables commute with H and are therefore constants of motion. How can we account for the flow of time in this situation? We would like to describe change in the system, but we can only use quantities which are constant on any trajectory. There seems to be a paradox.
A way out of this is given by the so-called *evolving constants*, first developed by Rovelli [12–14]. As noticed in the previous paragraph, even when we have a Newtonian time in the system, we can work with constants of motion parametrized by a real parameter t, representing the fixed instant of time at which we are looking at the system. We can adopt the same strategy here. Let's define a function τ on the constraint surface Σ, requiring every constant hypersurface to cut all trajectories once and only once. Assuming we can do so, evolution can be described by families of complete observables indexed by the parameter τ. Let F be a partial observable. We can generate a family of complete observables $\{F_T, T \in \mathbb{R}\}$ defining F_T to take a specific value of F in any point of the phase space. Namely, for every motion $\tilde{x}$, we assign to $F_T\,(\tilde{x})$ the value taken by F at the unique intersection of $\tilde{x}$ and the hypersurface $\tau = T$. As a function on $T^*\mathcal{C}$ [4], F_T commutes with H and defines a complete observable. For any point $x \in \Sigma$, $F_T\,(x)$ is interpreted as the value of F at "time" $\tau = T$ for the motion containing x. Hence, the set $\{F_T, T \in \mathbb{R}\}$ captures the evolution of the partial observable F with respect to the time parameter τ.
Note that this construction is not always possible.[5] The existence of a good time function τ is a necessary condition for this. However, the formalism remains consistant without time interpretation, as advocated by Rovelli [11]. In this respect, the presymplectic formalism is more general than the usual Hamiltonian theory, since it can accommodate timeless systems.

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[3] They have to to be gauge independent. Functions which don't commute with the constraint depend on unphysical degrees of freedom, and are not well-defined on the phase space Γ.

[4] obtained by analytical continuation for instance.

[5] For instance, if the constraint surface is a 2-sphere, we can't find such a function. An example of this is given in [3, 13]

II. REVIEW OF ISSUES AND RESULTS AT THE QUANTUM LEVEL

At the classical level, we have sketched how to solve the problem of time using presymplectic tools. First it appeared that the contradiction between constraint equation and change is only apparent, in the sense that any classical system (and in particular those with a Newtonian time) can be defined by a set of constraints. Then, we showed how to give a time interpretation to particular systems : those for which we can define families of complete observables accounting for the flow of time. Consequently, we can recover ordinary classical physics, with dynamics implemented through a time evolution. However, this interpretation is not essential, and we can easily imagine well-defined dynamical systems for which we cannot give a time interpretation using evolving constants. At this point, some authors (especially Rovelli [11]) are therefore tempted to deny the fundamental nature of time, and search for a completely timeless description of the world. However this idea doesn't really make consensus. This section is designed to give an overview of these issues at the quantum levels, where the evolving constant treatment is not anymore straightforward.

A. Quantization

We begin with an overview of the quantization procedure used in the context of LQG [16, 17], known as the *Dirac program*. As in the previous section, we'll focus on a finite-dimensional system with only one constraint, and we'll keep the same notation. A typical examples of application are cosmological models, for which we'll have to complete the following steps :

- Kinematical structure : find a representation of the classical partial observables on a kinematical Hilbert space $\mathcal{H}_{kin}$ satisfying the standard commutation relations (i.e. the correspondence $\{\ ,\ \} \longleftrightarrow -i/\hbar\,[\ ,\ ]$). Typically, we can choose $\mathcal{H}_{kin} = L_2\,(\mathcal{C})$.
- Promote the constraint H to an operator $\widehat{H}$ in $\mathcal{H}_{kin}$.
- Find the solutions of the constraint, i.e. the states Φ annihilated by $\widehat{H}$ (called physical states). They are usually non-normalizable, and do not belong to $\mathcal{H}_{kin}$. We can however define them as distributions on a proper dense subspace $\mathcal{S}$ of $\mathcal{H}_{kin}$.
- Physical inner product : since the solutions of the constraints are not normalizable with respect to the kinematical structure, we need to define a proper inner product on the space of the physical states. There is a general procedure, known as the group averaging method, leading to a unique physical scalar product. Alternatively, we can also look for an inner product with respect to which kinematical observables commuting with $\widehat{H}$ are self-adjoint. We call the resulting Hilbert space the physical Hilbert space, denoted by $\mathcal{H}_{phys}$.
- Find a complete set of classical complete observables and promote them to self-adjoint operators on $\mathcal{H}_{phys}$.

If we managed to complete this program for GR, we would get a fully *background independent* quantum theory of gravity. This quantization scheme is designed for gauge theories, and doesn't need any notion of spacetime background. In this sense, it is expected to give a

quantum theory of gravity implementing the very lessons of GR, and especially diffeomorphism invariance.
At this point, a lot of work has been done, but many issues remain. The kinematical structure of the theory is now well-known, and produced spectacular results : quantization of areas and volumes, recovery of the Hawking's formula for black hole entropy. However, despite some strong results [18], people are still fighting with the dynamical structure of the theory. That's why most of the work is now focused on path integral formulations of the theory, known as *spin foams*, or on symmetry reduced models like LQC. For the latter, the idea is to find out how to deal with the Hamiltonian constraint in simplified models, and try then to extend the solution to the full theory. As Abhay Ashtekar puts it [5], this strategy has been proved very successful in the past : this has for instance been the path followed in the early 20th century, from Bohr's atom to the general framework of quantum mechanics.

B. Some fundamental questions related to time in quantum gravity

For the sake of clarity, let's review some aspects of the problem of time in quantum gravity. Most of them emerge from interpretation issues, and especially conflicts with the Copenhagen interpretation of quantum mechanics. We closely follow a classification from Thiemann's book [4], focusing only on the specific points we are interested in :

- *No time problem*
  In ordinary quantum mechanics, the Hamiltonian is the generator of time evolution. In quantum gravity, we don't have a true Hamiltonian, since Hamilton's constraint is not unitary. We have a priori no time in the system.

- *Closed system problem*
  This issue emerges in cosmological contexts, and is therefore of first importance for LQC. First, if we want to study the whole universe, there is by definition no possible splitting of it in a system plus a measurement apparatus. Both should be described by the same theory, which strikingly contradicts the Copenhagen interpretation. (cite Rovelli's relational qm). The second aspect of the problem comes from difficulties with interpreting probabilities in this context. Since, again by definition, we can't prepare statistical ensembles of universes (cite D'Espagnat), it's not easy to give a meaning to expectation values and probabilities. We can see this as the problem of interpreting the *wave function of the universe.*

- *Measurement problem*
  Without time parameter, there is no straightforward way of ordering events. It's therefore hard to compute probabilities for sequences of measurements. Indeed, if we 'successively' measure non-commuting observables, what we ordinary need in quantum mechanics is to specify the times at which the measurements are performed. The collapses of the wave function are then ordered in this way. Without time, there is an ordering ambiguity, and the probabilities amplitudes depend on an arbitrary choice. Moreover, in quantum gravity complete observables are expected to be non-local, which seems to strengthen the difficulty. In this respect, the measurement problem appears to be even worse in quantum gravity than in ordinary quantum mechanics.

In the following, will try to present some ways out of the first problem, in the simplest cases. The two other issues are too hard to be deeply discussed here, but they'll appear sometimes in the next sections. We should also mention that as far as we know, they have not focused much work in LQC for the moment. The reason is that : first, there are many technical issues, and progress is fast on this side; secondly, a field can be used as an emergent time, which allows to build semi-classical states at late 'times', and compare the theory with classical cosmology without any reference to measurement procedures. [8]

C. Internal time

The simplest way of solving the problem of time is to mimic what we've done at the classical. As we saw, any well-behaved function on the constraint surface (i.e whose constant value surfaces cut any solution of Hamilton's equations once and only once) can be used as a time parameter. Straightforwardly extending the idea to the quantum theory would amount to quantize all the variables of the classical system except this specific parameter. However, since there is no reason for a time function to be unique at the classical level, this should bring some arbitrariness in the quantization scheme. For example, we could argue that in specific Newtonian systems, space variables evolve monotonically in time, and are therefore good time parameters. But in quantum mechanics, the Newtonian time is very different from space : it's not an observable, and it parametrizes the unitary evolution of the system. What does make it so singular, and what would happen if we tried to describe the quantum evolution with another classical variable? We'll discuss these questions in the third section, in the context of the dynamics of a free non relativistic particle.
What we would like to evoke now is the emergence of an internal time from the dynamical structure of a theory quantized following the Dirac procedure. We can find good examples of such a situation in homogeneous and isotropic cosmology with one scalar field ϕ. The configuration variables are ϕ and the scale factor a. To stick with LQC variables, will use a quantity called v instead, which is proportional to the volume of a fixed elementary cell of the universe (see [9]). The classical constraint is then :

$$C \equiv 12\pi G\left(vp_v\right)^2 - p_\phi^2\,, \tag{2.1}$$

where p_ϕ and p_v are the conjugate momenta of ϕ and v respectively. Using the canonical symplectic structure, it's not difficult to write down Hamilton's equations and compute their solutions. We can then show that ϕ is monotonic along any trajectory and is therefore a good time function at the classical level.
Now there are two ways of quantizing the system : one corresponds to the traditional method leading to the Wheeler-DeWitt theory (WdW), and the second leads to LQC. The differences between come from the geometric part of the constraint. In both theories, matter is quantized through the usual procedure : ϕ is represented as a multiplicative operator and p_ϕ as $-i\hbar\partial_\phi$, on $L_2\left(\mathbb{R}, d\phi\right)$. As for the geometry, we either use the same idea and get the WdW theory, or a loop representation which mimics what has been done in LQG. In this latter case, we end up with LQC, which predicts large deviations from the WdW dynamics at the Planck scale. We won't go through the details, since in both theories, the quantized constraint is of the following form :

$$\hat{C} \equiv -\hat{\theta} - {\partial_\phi}^2\,, \tag{2.2}$$

where $\hat{\theta}$ is a positive self-adjoint operator on a kinematical Hilbert space $\mathcal{H}_{kin}$. Vectors in $\mathcal{H}_{kin}$ can be written as wave functions $\Psi\left(v,\phi\right)$. $\hat{\theta}$ is the differential operator $12\pi G\left(v\partial_v\right)^2$ in the WdW theory, and a difference operator in LQC [9]. Anyway, since both are self-adjoint and positive on their respective Hilbert spaces, their square roots are well defined and we can look for positive frequency solutions verifying :

$$-\mathrm{i}\partial_\phi \Psi(v,\phi) = \sqrt{\hat{\theta}}\Psi(v,\phi)\ . \tag{2.3}$$

We therefore end up with the Schrodinger equation corresponding to the Hamiltonian $\sqrt{\hat{\theta}}$. Moreover, the group averaging procedure allows to uniquely deduce the physical inner product from the kinematical structure. The result gives to ϕ the status of Newtonian time of the system : given two physical states Ψ_1 and Ψ_2, their physical inner product only depends on their values at any fixed value ϕ^0 of the scalar field. $\sqrt{\hat{\theta}}$ is obviously unitary with respect to this inner product, and ϕ can fully be interpreted as a time.
Here, we see that the time interpretation of ϕ truly emerges from general quantization procedures : the Dirac program as the general framework, and averaging methods to go from the kinematical to the dynamical structures. It's really convenient since we can then use the formalism of ordinary quantum mechanics. The no time problem is solved in an elegant way, and as we already pointed it, this allows to build semi-classical states and work in a regime where the other interpretational issues play no role. [7–9]
Unfortunately, we don't expect any preferred time to come up in the full theory, so we don't see how this idea could play a role in LQG. Even in small generalizations of the model, where we may have more than one consistent choice of internal time, it's not clear whether the final theory would be dependent on this choice or not. We'll discuss this aspect in the fourth section, which deals with a LQC model with more than one scalar field. Despite these severe drawbacks, it's in our view worth noting that time can naturally emerge from a timeless theory.

D. Evolving constants and conditional probabilities

We now would like to discuss the evolving constants program and point out some technical and interpretational issues. Consider a system admitting a good time function τ, and suppose we are able to represent the classical evolving constants on the physical Hilbert space, so that they form the complete set of observables of the last step of the Dirac quantization scheme. This situation is more general than the one of the previous paragraph, since we don't require the constraint to be equivalent to a Schrodinger equation. We could for example have deviations from a unitary evolution, and recover the Schrodinger picture in a certain limit only. [12] However, this is enough to account for the *illusion of change* [10] we experiment in our everyday life : all our observations are parametrized by a one dimensional real quantity T (the value of τ).
Now it's time to recall the main motivation of the construction : we want physical quantities to be gauge invariant. That's why we want to use evolving constants : they are complete observables. Troubles come then from the interpretation of the time parameter T. The physical quantities of the system are families of operators $\left\{\widehat{F_T}, T\in\mathbb{R}\right\}$, indexed by T. They code the information about *the measurement of the value of F at time T* : possible outcomes, transition amplitudes. So all our predictions are now indexed by a quantity which is not itself gauge invariant. We may wonder how in practice we could perform such measurements, since we supposed from the start that all we can observe is gauge invariant.

Another disconcerted aspect of this construction is that T remains a scalar quantity after quantization. At the classical level it was a coordinate and had the same status as other phase space quantities, but it's the only one which has not been turned into an operator.
All this leads to question the relevance of the evolving constants framework. In our view, we cannot be satisfied at this point, and what we need is to relate the time parameter T to an operator. This is what Gambini, Pullin, Porto and collaborators propose to do, through conditional probabilities [19]. In the context of the measurement problem and decoherence, they pointed out that every observed quantities should be defined quantum mechanically, and therefore related to operators [20, 21]. Even at the level of non-relativistic quantum mechanics, they claim that the Newtonian time t should have a limited physical meaning. In any experiment we should consider the quantum nature of real clocks, and give up the absolute time t. In our context, the time parameter T would be *measured* by a quantum clock, corresponding to the evolving observable $\widehat{t}(T)$. Now, instead of information about the measurement of the value of F at time T, the theory should only predict probabilities of measuring the value F of $\widehat{F}$ knowing that the quantum clock $\widehat{t}$ takes the value t. Gambini, Porto and Pullin give a formula for such a probability $P\left(\widehat{F} \to F | \widehat{t} \to t\right)$:

$$P\left(\widehat{F} \to F | \widehat{t} \to t\right) = \lim_{\tau \to \infty} \frac{\int_{-\tau}^{\tau} dT \mathrm{Tr}\left(P_F(T) P_t(T) \rho P_t(T)\right)}{\int_{-\tau}^{\tau} dT \mathrm{Tr}\left(P_t(T) \rho\right)}, \tag{2.4}$$

where ρ is the density matrix of the system, and $P_F(T)$ (resp. $P_t(T)$) is the projector on the eigenspace of $\widehat{F}_T$ (resp. $\widehat{t}_T$) with eigenvalue F (resp. t). We see that T is treated as an unobservable quantity and integrated out. We thus obtain a gauge invariant quantity which is independent of T. This is a lot more satisfactory, and the physical interpretation is clearer.

E. Partial observables

Another possibility to solve the issues related to the evolving constants program has been advocated by Carlo Rovelli [11, 15]. It emphasizes the role played by partial observables, and explicitly makes a distinction between kinematical and dynamical considerations. At the moment, we have focused on the notion of complete observable, and defined the partial observables as quantities which are not gauge invariant. We need now to be more precise. In Rovelli's words, a partial observable is *a physical quantity to which we can associate a measuring procedure leading to a number*. With this definition, time and space are partial observables in non-relativistic physics. Now, in the spirit of the previous discussions on gauge invariance, a complete observable is, in simpler words, *a quantity whose probability distribution can be predicted by the theory*. So, as we already noticed, the position of a non-relativistic particle in function of the Newtonian time is a complete observable. But position or time alone are not.
Rovelli's claim is that partial observables should be associated with kinematics, and complete observables with dynamics. In quantum theory, this leads to emphasize the role played by the kinematical Hilbert space. Since we noted that in classical physics, outcomes of measurements are related to partial observables, we can suppose this to remain true in quantum physics. In this perspective, we should not see the kinematical Hilbert space as a simple step towards the quantized theory. On the contrary we want to use it to determine the possible

outcomes of the theory. The natural way of doing so is to assume that possible outcomes of measurement are determine by spectra of *kinematical* observables, which don't need to commute with the constraint $\widehat{H}$.[6]
Let's briefly sketch the general framework emerging from these considerations [15]. A quantum system is now a triplet $(\mathcal{H}_{kin}, \mathcal{H}_{phys}, \mathbb{P})$, where $\mathbb{P}$ is a linear function from $\mathcal{H}_{kin}$ to $\mathcal{H}_{phys}$, associating a solution to the constraint to any kinematical state. $\mathbb{P}$ can be defined through group averaging methods, and is formally given by $\int dt \exp(-it\widehat{H})$. The possible outcomes of measurements are given by the spectra of the kinematical observables we want to measure. Dynamics, or transition amplitudes are then computed in the physical Hilbert space. As an example, let's consider a particle in a one-dimensional space. Partial observables are its position x and time t. They correspond to kinematical operators $\widehat{X}$ and $\widehat{T}$, acting on $\mathcal{H}_{kin}$. They can be diagonalized, and we note their common eigenstates $|x,t\rangle$, where x (resp. t) lies in the spectrum of $\widehat{X}$ (resp. $\widehat{X}$). Since complete observables are correlations between position and time, the whole dynamics can be coded in transition amplitudes between kinematical states $|x,t\rangle$. Let's denote $|\mathbb{P}\, x,t\rangle$ the image of $|x,t\rangle$ by $\mathbb{P}$. Then the transition amplitude between two states $|x,t\rangle$ and $|x',t'\rangle$ is simply given by $\langle \mathbb{P}\, x,t | \mathbb{P}\, x,t\rangle_{phys}$, where $\langle \, | \, \rangle_{phys}$ is the physical inner product, and the states are supposed to be normalized with respect to it (and not the kinematical inner product).
This formalism differs from the evolving constants framework, but both reduce to standard quantum mechanics for conventional systems for which the constraint is of the form $H = p_t + h$. In this case, the evolving constants formulation would correspond to the Heisenberg picture of the theory, while what we just described would be the Schrodinger picture. However, as pointed out in [22, 24], the two descriptions are not always compatible, and can therefore be seen as alternative quantizations of the same classical model.
As for the fundamental problems we listed at the beginning of this section, the partial observables point of view solves the no time issue by relaxing the conditions on a time variable : it's only a partial observable, like any other. Thus the standard notion of time is given up, and the theory is essentially timeless. There are also hints on how the two other issues could be solved. Since now time as no particular status, we have lost the corresponding events ordering, and the collapse issue is challenging. One solution has been provided in [25] : once more (remember the preceding paragraph about conditional probabilities), it consists in taking into account the quantum nature of the measuring apparatus by working with the joint system + apparatus system.

III. POSITION AS A TIME FUNCTION FOR A NON-RELATIVISTIC FREE PARTICLE, AND CONDITIONAL PROBABILITIES

In this section, we would like to discuss a very simple system for which we may have more than one possible choice of time function at the classical level : the non-relativistic free particle. The question is then : what makes the Newtonian time so special, and can we use these alternative time functions in the quantum theory? In the past, some authors

[6] In LQG, this point of view rules out debates about the physical nature of area and volume spectra predicted by the theory [22]. In fact, this prediction [23] was made at the kinematical level only, and it's not clear at this point whether the corresponding physical observables will remain quantized [24]. But if possible outcomes are determined by kinematics, this point is not relevant.

[26] have been interested in this. Here, we would like to give a new perspective on the issue, using conditional probabilities. For simplicity, we'll develop our ideas in one dimension of space, and give the generalization to any dimension at the end of the section.

A. Classical system and good time functions

Using the notations introduced in the previous sections, the configuration space $\mathcal{C}$ of the theory is coordinatized by the position of the particle x and time x^0. We note their respective conjugate momenta p and p_0, and $T^*\mathcal{C}$ is equipped with the canonical symplectic structure. The dynamics is coded by a constraint $H \equiv p_0 + \frac{p^2}{2m}$, where m is the mass of the particle. Now, it's easy to show that p and $q \equiv x - \frac{x^0 p}{m}$ commute with the constraint, and are therefore complete observables.
As it is well known, any solution $x(x^0)$ is linear in x^0. Thus x can be used as a time on almost every trajectories, i.e. as long as p is non zero. The phase space can be split in two parts, and on each of them x is a good classical clock : constant x hypersurfaces cut any trajectory once and only once. Other well behaved time functions can be built (see [26]), but we'll consider only x in the following. Our purpose is to address the closed system problem, and use the non relativistic particle as a toy model. In this context, we don't have access to an external clock, so it seems natural to use evolving constants. And for the reasons evoked before, we'll try to formulate the dynamics in terms of conditional probabilities only. Since we have at least two good clocks at the classical level, we can use one of them as a parameter for the evolving constants, and the other as a quantum clock. Finally, we would like to be able to compare the results with standard quantum mechanics, and therefore end up with conditional probabilities involving the Newtonian time x^0. That's why we'll build evolving constants parametrized by the position x.

B. Quantization

Like in [27], we choose to work in the momentum representation and define $\mathcal{H}_{kin}$ as $L^2(\mathbb{R}^2, dp_0 dp)$. Since solutions to the constraint are not normalizable, we actually need a Gelfand triple $\mathcal{S} \subset \mathcal{H}_{kin} \subset \mathcal{S}^*$, where $\mathcal{S}$ is a dense subset of $\mathcal{H}_{kin}$. We choose $\mathcal{S}$ to be the Schwartz space. The variables of $T^*\mathcal{C}$ are represented on $\mathcal{H}_{kin}$ by the following operators (from now, we set $\hbar = 1$) :

$$\widehat{p} = p \quad , \quad \widehat{x} = \mathrm{i}\partial_p \, , \tag{3.1}$$

$$\widehat{p_0} = p_0 \quad , \quad \widehat{x^0} = \mathrm{i}\partial_{p_0} \, . \tag{3.2}$$

Physical states are solutions $\Psi(p, p_0)$ of the equation $p_0 + \frac{p^2}{2m}\Psi(p, p_0) = 0$:

$$\Psi(p, p_0) = \delta\left(p_0 + \frac{p^2}{2m}\right)\tilde{\Psi}(p) \in \mathcal{S}^* \, . \tag{3.3}$$

Finally, using averaging methods, the physical inner product between two states Ψ_1 and Ψ_2 is shown to be the standard one :

$$\langle\Psi_1|\Psi_2\rangle = \int dp_0 dp \delta\left(p_0+\frac{p^2}{2m}\right)\overline{\tilde{\Psi}_1(p)}\tilde{\Psi}_2(p) \tag{3.4}$$

$$= \int dp \overline{\tilde{\Psi}_1(p)}\tilde{\Psi}_2(p)\,. \tag{3.5}$$

C. Choice of evolving constants

We want to use x as a time function, so we'll note T the values taken by this function along any trajectory of the system. Classically we can compute the function $x^0(T)$ corresponding to the value of x^0 at "time" T :

$$x^0(T) = \frac{m}{p}(T-q)\ , \tag{3.6}$$

which commute with the constraint and is singular in $p=0$. Since p and q do not commute, there are ordering ambiguities for the operator $\widehat{x^0(T)}$. Following cite (Rovelli), we choose a symmetric ordering[7] :

$$\widehat{x^0(T)} = m\frac{1}{\sqrt{p}}(T-\mathrm{i}\partial_p)\frac{1}{\sqrt{p}} + \mathrm{i}\partial_{p_0}\,. \tag{3.7}$$

Since it's a complete observable, $\widehat{x^0(T)}$ doesn't act on the δ part of the physical states. We can therefore define its action on $\tilde{\Psi}(p)$ functions and forget the derivative with respect to p_0. As enlightened in [27], this operator is symmetric, but not self-adjoint and does not admit self-adjoint extensions. To see it, we can use a simple method from spectral theory [29]: $\mathcal{K}_- \equiv \ker(\widehat{\mathrm{x^0(T)}}-\mathrm{i})$ and $\mathcal{K}_+ \equiv \ker(\widehat{\mathrm{x^0(T)}}+\mathrm{i})$ must have the same dimensionality to at least admit self-adjoint extensions. If this dimension is zero, $\widehat{x^0(T)}$ is self-adjoint. Here we can show that $\dim(\mathcal{K}_-)=0$ and $\dim(\mathcal{K}_+)=1$, which proves that $\widehat{x^0(T)}$ is not self-adjoint. This being said, it seems hard to assign probabilities to $\widehat{x^0(T)}$, since the spectral theorem cannot be used in this case. This is the standard argument against the possibility of defining a time operator in quantum mechanics. In the following we will adopt another point of view : as done in (cite Rovelli etc.), we are going to use a family of self-adjoint operators arbitrary close to $\widehat{x^0(T)}$ (for the weak topology). Put it differently, we want to go through a regulation.

D. Regulation of the time operator

Following [28], let's define a regulated operator $\widehat{x^0(T)}_\epsilon$ for any $\epsilon>0$:

$$\widehat{x^0(T)}_\epsilon = m\sqrt{f_\epsilon(p)}\,(T-\mathrm{i}\partial_p)\sqrt{f_\epsilon(p)}\,, \tag{3.8}$$

where

$$f_\epsilon(p) = \begin{cases} 1/p & \text{if p} > \epsilon \\ \epsilon^{-2}p & \text{if p} < \epsilon \end{cases}\,. \tag{3.9}$$

[7] when $p<0$, we define $\sqrt{p}=\mathrm{i}\sqrt{-p}$

Its eigenvectors can be shown to be indexed by a discrete parameter $\eta \in \{+,-\}$ together with the eigenvalues $x^0 \in \mathbb{R}$. We not them $\Psi^\eta_{\epsilon,T,x^0}$, and the following relations hold :

$$\Psi^\eta_{\epsilon,T,x^0}(p) = \frac{\theta(\eta p)}{\sqrt{2\pi m f_\epsilon(p)}} \exp\left(\mathrm{i}\frac{x^0}{m}\int_{\eta\epsilon}^{p}\frac{dk}{f_\epsilon(k)}\right)\exp(-\mathrm{i}p) \ , \tag{3.10}$$

$$\left\langle \Psi^\eta_{\epsilon,T,x^0} | \Psi^{\eta'}_{\epsilon,T,x^{0\prime}} \right\rangle = \delta^{\eta,\eta'}\delta\left(x^0 - x^{0\prime}\right) \ , \tag{3.11}$$

$$\sum_{\eta=\pm}\int_{-\infty}^{+\infty} dx^0 \overline{\Psi^\eta_{\epsilon,T,x^0}(p)}\Psi^\eta_{\epsilon,T,x^0}(p') = \delta(p-p') \ . \tag{3.12}$$

We have an explicit spectral decomposition of $\widehat{x^0(T)}_\epsilon$, which is therefore self-adjoint. Moreover, since $\widehat{x^0(T)}_\epsilon$ converges pointwise to $\widehat{x^0(T)}$ when $\epsilon \to 0$, it is reasonable to think that the probabilities we are interested in can be computed by doing all the calculations with $\widehat{x^0(T)}_\epsilon$, and ultimately taking the limit $\epsilon \to 0$.

E. Conditional probabilities

We now have all the material needed to compute the probability of measuring a certain value of the momentum p under the condition that the value x^0 of the Newtonian time is observed. Since the spectra are continuous, we can in fact only ask for p and x to be in intervals of the form $[p_1,p_2]$ and $[x^0 - \Delta x^0/2, x^0 + \Delta x^0/2]$. The corresponding projectors are :

$$P_{[p_1,p_2]}(T) = \int_{p_1}^{p_2} dk\delta(p-k) \ , \tag{3.13}$$

$$P_{x^0,\Delta x^0}(T) = \sum_{\eta=\pm}\int_{x^0-\Delta x^0/2}^{x^0+\Delta x^0/2} dt \left|\Psi^\eta_{\epsilon,T,t}\right\rangle\left\langle\Psi^\eta_{\epsilon,T,t}\right| \ . \tag{3.14}$$

We consider a pure state associated to a wave function $\Psi(p)$.[8] We don't expect to find reasonable results when its domain is not included in one half of the real line, because even at the classical level, x^0 is a good clock on one half of the constraint surface only. For definiteness, we'll therefore suppose that Ψ has support on $\mathbb{R}^*_+$. For the same reason, we recquire $[p_1,p_2] \subset \mathbb{R}^*_+$.
Following (cite G P P), the conditional probabilities we would like to compute is :

$$P\left([p_1,p_2]\,|x^0,\Delta x^0\right) = \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{\int_{-\tau}^{\tau} dT \mathrm{Tr}\left(P_{[p_1,p_2]}(T)P_{x^0,\Delta x^0}(T)\rho P_{x^0,\Delta x^0}(T)\right)}{\int_{-\tau}^{\tau} dT \mathrm{Tr}\left(P_{x^0,\Delta x^0}(T)\rho\right)} \ . \tag{3.15}$$

Despite the apparent complexity of this expression, we can successively take the two limits, which leads to the nice following equality :

$$P\left([p_1,p_2]\,|x^0,\Delta x^0\right) = \frac{\int_0^\infty dk k|\Psi(k)|^2 F(k)}{\int_0^\infty dk k|\Psi(k)|^2} \ , \tag{3.16}$$

[8] In this case $\langle p|\rho|p'\rangle = \Psi(p)\overline{\Psi(p')}$

with :

$$F(k) = \frac{1}{\pi} \int_{\frac{{p_1}^2 - k^2}{4m\hbar}\Delta x^0}^{\frac{{p_2}^2 - k^2}{4m\hbar}\Delta x^0} dX \left(\frac{\sin(X)}{X} \right)^2 . \tag{3.17}$$

For the coming discussion, we have reintroduced the $\hbar$ dependence.

F. Discussion

Let's now compare our result with the standard probabilities, and try to find a limit in which both predictions coincide. The expression (3.16) looks relatively similar to the probability of measuring p in $[p_1, p_2]$ at time t we would get from ordinary quantum mechanics :

$$\frac{\int_{p_1}^{p_2} dk |\Psi(k)|^2}{\int_0^\infty dk |\Psi(k)|^2} . \tag{3.18}$$

One main difference is that in our case, time is not anymore considered as an absolute quantity, and is subject to quantum fluctuations. Another one is the measure of integration : in ordinary quantum mechanics we use the Lebesgue measure dk, which is replaced in (3.16) by kdk. As for the form factor $F(k)$, we could actually see it as a function coding the quantum fluctuations of the clock. The reason for this is that, because the integral over $\mathbb{R}$ of the squared sine cardinal function is π, and the domains which are far from the origin poorly contribute to it, $F(k)$ converges (weakly) to the characteristic function of $[p_1, p_2]$ when $\frac{{p_2}^2 - {p_1}^2}{4m\hbar}\Delta x^0 \to +\infty$. We have therefore :

$$P\left([p_1, p_2] | x^0, \Delta x^0\right) \approx_{\frac{{p_2}^2 - {p_1}^2}{4m}\Delta x^0 \gg \hbar} \frac{\int_{p_1}^{p_2} dkk |\Psi(k)|^2}{\int_0^\infty dkk |\Psi(k)|^2} . \tag{3.19}$$

The form of the criterion $\frac{{p_2}^2 - {p_1}^2}{4m}\Delta x^0 \gg \hbar$ suggests to draw a link with time-energy uncertainty relations and give the following interpretation : the quantum fluctuations of the clock can be neglected when the bounds on the measurement are such that the product of the uncertainty on the energy of the particle and the uncertainty on time is a lot greater than its minimum value. Actually, by examining the case of a particle moving in more than one dimension, we can show that only the energy along the space dimension which has been used as a parameter-time enters the criterion. So what plays a role is not the total energy of the particle, but really that of the clock. That's why we think we should relate the form factor F to the fluctuations of the quantum clock.

We would now like to understand the extra k factor in the measure. Physically, we expect the position of the particle to be a good clock only for coherent states. For states having support on large intervals in momentum space, the corresponding wave function in the position representation should rapidly spread. Consequently, x and t would not define the same notion of simultaneity : components of the wave function $\Psi(p)$ corresponding to a same position at $t = t_0$ could be far appart at $t = t_1 > t_0$. Quantitatively, if the spread Δp is small compared to the mean value $\overline{p}$ of the momentum, then (3.16) approximately equals the standard expression (3.18).

To conclude with this section, we see that contrary to what is generally claimed, we can in certain conditions associate a self-adjoint operator to time measurements. This allows to describe true quantum measurements, where clock fluctuations show up. Our study suggests that their effects could be linked to time-energy uncertainty relations. We would also like to emphasize that since we used the non-relativistic particle as a toy cosmological model, we didn't expect to recover standard quantum mechanics exactly. In cosmology, no external clock can be used and departures from ordinary quantum theory might be generic : since we don't have the possibility of choosing arbitrary precise external clocks, unitary could be effectively lost. In [20, 21], intrinsic limitations to real clocks have been pointed out, which lead to a fundamental mechanism of decoherence and effective loss of unitarity. We think that the argument is even stronger in cosmology. Essentially by definition, the clock is part of the universe, and not an auxiliary device weakly interacting with the system under study.

IV. MULTIPLE CHOICES OF TIME IN A SIMPLE COSMOLOGICAL MODEL

In this section, we consider a model of homogeneous and isotropic spacetimes. In the LQC models studied in the literature [7–9], there is a unique scalar field ϕ, which can be viewed as an emergent time. We would like to understand what happens when matter does not only consist in one scalar field. Does one of the fields still be used as a time? If yes, are the different choices consistent with each other? For definiteness and simplicity, we assume a universe filled with three scalar fields, and adopt a WdW quantization.

A. Kinematical structure

We use the variables and notations (briefly introduced before) from [9], in which the so called *improved dynamics* of LQC is constructed. The WdW theory is also discussed, and we especially refer to this part of the article.
At the kinematical level, a state of the system is a function $\Psi \in L^2_s(\mathbb{R}^4, \frac{1}{v}dvd\phi_1 d\phi_2 d\phi_3)$, where the subscript s means that Ψ is symmetric in v. Recalling that v represents the oriented volume of an elementary cell of the universe, this condition amount to consider the two orientations as physically equivalent. We will alternatively use the momentum representation, defined on the Hilbert space $L_2(\mathbb{R}^4, dkd\sigma_1 d\sigma_2 d\sigma_3)$. $\{k, \sigma_1, \sigma_2, \sigma_3\}$ are respectively conjugated to $\{\ln|v|, \phi_1, \phi_2, \phi_3\}$, and defined by :

$$\Psi(v, \phi_1, \phi_2, \phi_3) = \int dkd\sigma_1 d\sigma_2 d\sigma_3 f(k, \sigma_1, \sigma_2, \sigma_3) e_k(v) e_{\sigma_1}(\phi_1) e_{\sigma_2}(\phi_2) e_{\sigma_3}(\phi_3) \quad (4.1)$$

$$e_k(v) = \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i}k \ln|v|) \quad (4.2)$$

$$e_{\sigma_i}(\phi_i) = \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i}\sigma_i \phi_i) \quad (4.3)$$

$$f \in L_2(\mathbb{R}^4, dkd\sigma_1 d\sigma_2 d\sigma_3) \quad (4.4)$$

The dynamics is given by the constraint $\hat{C}$:

$$\hat{C} = 12\pi G v \partial_v v \partial_v - \sum_{i \in \{1,2,3\}} \frac{\partial^2 \Psi}{\partial {\phi_i}^2} \quad (4.5)$$

B. Physical Hilbert space

The action of $\hat{C}$ on a state Ψ is :

$$\hat{C}\Psi(v,\phi_1,\phi_2,\phi_3) = \int dkd\sigma_1 d\sigma_2 d\sigma_3 f(k,\sigma_1,\sigma_2,\sigma_3)\hat{C}e_k(v)e_{\sigma_1}(\phi_1)e_{\sigma_2}(\phi_2)e_{\sigma_3}(\phi_3) \quad (4.6)$$

$$= \int dkd\sigma_1 d\sigma_2 d\sigma_3 f(k,\sigma_1,\sigma_2,\sigma_3)\left(12\pi Gk^2 - \sum_{i\in\{1,2,3\}} {\sigma_i}^2\right) \quad (4.7)$$

$$\times\, e_k(v)e_{\sigma_1}(\phi_1)e_{\sigma_2}(\phi_2)e_{\sigma_3}(\phi_3)\,. \quad (4.8)$$

We see then that in the momentum representation, the physical states must verify :

$$\left(12\pi Gk^2 - \sum_{i\in\{1,2,3\}} {\sigma_i}^2\right) f(k,\sigma_1,\sigma_2,\sigma_3) = 0\,. \quad (4.9)$$

As usual, these states are non-normalizable. They are distributions on the Schwartz space of $L_2(\mathbb{R}^4, dkd\sigma_1 d\sigma_2 d\sigma_3)$:

$$f(k,\sigma_1,\sigma_2,\sigma_3) = \delta\left(12\pi Gk^2 - \sum_{i\in\{1,2,3\}} {\sigma_i}^2\right)\tilde{f}(k,\sigma_1,\sigma_2,\sigma_3)\,. \quad (4.10)$$

Finally, the group averaging procedure gives the physical scalar product between two physical states f_1 and f_2 :

$$\langle f_1|f_2\rangle = \int dkd\sigma_1 d\sigma_2 d\sigma_3 \delta\left(12\pi Gk^2 - \sum_i {\sigma_i}^2\right)\overline{\tilde{f}_1(k,\sigma_1,\sigma_2,\sigma_3)}\tilde{f}_2(k,\sigma_1,\sigma_2,\sigma_3)\,. \quad (4.11)$$

C. Quantized observables

On the kinematical Hilbert space $L_{2s}(\mathbb{R}^4, \frac{1}{v}dvd\phi_1 d\phi_2 d\phi_3)$, we define the observables $\widehat{|v|}$, $\widehat{\phi_i}$ as multiplicative operators. The operators corresponding to the classical momenta p_{ϕ_i} are then $\widehat{p_{\phi_i}} \equiv -\mathrm{i}\hbar\partial_{\phi_i}$. We can easily show that the corresponding actions in the momentum representation defined before are :

$$\widehat{|v|}f(k,\sigma_1,\sigma_2,\sigma_3) = \exp\left(\mathrm{i}\frac{\partial}{\partial k}\right)f(k,\sigma_1,\sigma_2,\sigma_3)\,, \quad (4.12)$$

$$\widehat{\phi_i}f(k,\sigma_1,\sigma_2,\sigma_3) = \mathrm{i}\frac{\partial}{\partial\sigma_i}f(k,\sigma_1,\sigma_2,\sigma_3)\,, \quad (4.13)$$

$$\widehat{p_{\phi_i}}f(k,\sigma_1,\sigma_2,\sigma_3) = \hbar\sigma_i f(k,\sigma_1,\sigma_2,\sigma_3)\,. \quad (4.14)$$

These operators are partial observables, and are not well defined on the physical Hilbert space. To build complete observables, we'll use the evolving constants strategy. At the classical level, any of the ϕ_i can be used as a relational time, and in particular ϕ_1. We first compute the three following families of complete observables in the classical theory : $|v|_{|\phi_1^0}$, $\phi_{2|\phi_1^0}$ and $\phi_{3|\phi_1^0}$. They can explicitly be written as :

$$|v|_{|\phi_1^0}(v, \phi_1, \phi_2, \phi_3) \;=\; |v| \exp\left(12\pi G v p_v \frac{\phi_1^0 - \phi_1}{p_{\phi_1}}\right) , \tag{4.15}$$

$$\phi_{i|\phi_1^0}(v, \phi_1, \phi_2, \phi_3) \;=\; \phi_i + p_{\phi_i} \frac{\phi_1^0 - \phi_1}{p_{\phi_1}} \text{ for i} \in \{2, 3\} . \tag{4.16}$$

With a similar symmetric ordering prescription as used for the temporal operator in the previous section, we quantize $\widehat{\phi}_{i|\phi_1^0}$:

$$\widehat{\phi}_{i|\phi_1^0} \;=\; \mathrm{i}\frac{\partial}{\partial\sigma_i} + \sigma_i \frac{1}{\sqrt{\sigma_1}}\left(\phi_1^0 - \mathrm{i}\frac{\partial}{\partial\sigma_1}\right)\frac{1}{\sqrt{\sigma_1}} \tag{4.17}$$

$$\;=\; \mathrm{i}\frac{\partial}{\partial\sigma_i} + \frac{\sigma_i}{\sigma_1}\left(\phi_1^0 - \mathrm{i}\frac{\partial}{\partial\sigma_1} + \mathrm{i}\frac{1}{2\sigma_1}\right) . \tag{4.18}$$

It is somewhat subtler to apply the same idea to $|v|_{|\phi_1^0}$. The issue is, while $\widehat{|v|}$ is well-defined on the kinematical Hilbert space (square integrable even functions in the position representation), neither $\widehat{v}$ nor $\widehat{p_v}$ are. However we will naturally assume that their product can be defined as $\widehat{vp_v} = -\mathrm{i}\hbar v \frac{\partial}{\partial v}$, which acts as multiplication by $\hbar k$ on $e_k\,(v)$. This leads to a nice expression of $\widehat{|v|}_{|\phi_1^0}$:

$$\widehat{|v|}_{|\phi_1^0} \;=\; \exp\left(\mathrm{i}\frac{\partial}{\partial k} + k\frac{1}{\sqrt{\sigma_1}}\left(\phi_1^0 - \mathrm{i}\frac{\partial}{\partial\sigma_1}\right)\frac{1}{\sqrt{\sigma_1}}\right) \tag{4.19}$$

$$\;=\; \exp\left(\mathrm{i}\frac{\partial}{\partial k} + \frac{k}{\sigma_1}\left(\phi_1^0 - \mathrm{i}\frac{\partial}{\partial\sigma_1} + \mathrm{i}\frac{1}{2\sigma_1}\right)\right) . \tag{4.20}$$

D. Self-adjointness

Like for the non-relativistic particle, we at this point need to wonder whether the observables we just defined are self-adjoint or not. If not, we'll have to find a way of regulating them. Let's use again the defect indices method. To do so, we'll compute the common eigenvectors of the observables. We will in particular get those corresponding to eigenvalues i and −i, in which we are interested right now. We thus begin with solving the system :

$$\left(\mathrm{i}\frac{\partial}{\partial\sigma_i} + \frac{\sigma_i}{\sigma_1}\left(\phi_1 - \mathrm{i}\frac{\partial}{\partial\sigma_1} + \mathrm{i}\frac{1}{2\sigma_1}\right) - \phi_i\right) f\,(k, \sigma_1, \sigma_2, \sigma_3) \;=\; 0\,,\ \text{for i} \in \{2, 3\} \tag{4.21}$$

$$\left(\mathrm{i}\frac{\partial}{\partial k} + \frac{k}{\sigma_1}\left(\phi_1 - \mathrm{i}\frac{\partial}{\partial\sigma_1} + \mathrm{i}\frac{1}{2\sigma_1}\right) - \lambda\right) f\,(k, \sigma_1, \sigma_2, \sigma_3) \;=\; 0 \tag{4.22}$$

with $(\phi_1, \phi_2, \phi_3, \lambda) \in \mathbb{R}^4$. The space of solutions is generated by the functions $\{f_{\phi_1,\phi_2,\phi_3,\lambda;\mu},\ \mu \in \mathbb{C}\}$, with :

$$f_{\phi_1,\phi_2,\phi_3,\lambda;\mu}\,(k, \sigma_1, \sigma_2, \sigma_3) \;\propto\; \sqrt{|\sigma_1|}\exp\left(-\mathrm{i}\left(\sum_{i\in\{1,2,3\}} \phi_i\sigma_i + \lambda k\right)\right) \tag{4.23}$$

$$\times \exp\left(-i\mu\left(12\pi G k^2 - \sum_{i\in\{1,2,3\}} {\sigma_i}^2\right)\right) . \tag{4.24}$$

Integrating μ over $\mathbb{R}$ yields the physical state we are interested in :

$$f_{\phi_1,\phi_2,\phi_3,\lambda}\left(k,\sigma_1,\sigma_2,\sigma_3\right) \propto \delta\left(12\pi G k^2 - \sum_{i\in\{1,2,3\}} {\sigma_i}^2\right) \tag{4.25}$$

$$\times\sqrt{|\sigma_1|}\exp\left(-\mathrm{i}\left(\sum_{i\in\{1,2,3\}}\phi_i\sigma_i + \lambda k\right)\right)\,. \tag{4.26}$$

For every $\phi_1 \in \mathbb{R}$, we thus obtain a family of physical states $\left\{\tilde{f}_{\phi_1,\phi_2,\phi_3,|v|}\right\}$, labelled by the eigenvalues ϕ_2, ϕ_3, and $|v|$ of $\phi_{2|\phi_1}$, $\phi_{3|\phi_1}$ and $|v|_{|\phi_1}$:

$$\tilde{f}_{\phi_1,\phi_2,\phi_3,|v|}\left(k,\sigma_1,\sigma_2,\sigma_3\right) \propto \sqrt{|\sigma_1|}\exp\left(-\mathrm{i}\left(\sum_{i\in\{1,2,3\}}\phi_i\sigma_i + \ln|v|k\right)\right)\,. \tag{4.27}$$

We can now see if, say $\widehat{\phi}_{2|\phi_1^0}$ is self-adjoint or not. Let's set the eigenvalue $\phi_2 = \pm\mathrm{i}$. The σ_2 dependence of $\tilde{f}_{\phi_1,\phi_2,\phi_3,|v|}$ becomes $e^{\pm\phi_2}$, and the state fails to be normalizable with respect to the physical inner product. It is therefore out of $\mathcal{H}_{phys}$, and the defect indices are both zero :

$$\dim\ker(\widehat{\phi}_{2|\phi_1^0} \pm \mathrm{i}) = 0\,. \tag{4.28}$$

We conclude that $\widehat{\phi}_{i|\phi_1^0}$ is self-adjoint. The result obviously applies to the two other observables, so we finally get a complete set of self-adjoint operators. This is good news, and a bit surprising, since this wasn't true for the non-relativistic particle. So we can expect to be able to easily write conditional probabilities. However, we didn't manage to do so at the moment. We'll therefore shortly review open issues and future work in a last paragraph.

E. Perspectives

The reason why we did not manage to compute conditional probabilities yet is purely technical. In the non-relativistic particle case, we took advantage of the presence of Dirac distributions in the orthogonality relations to reduce the expressions to single integrals. Unfortunately, here the orthogonality relations are more complicated. The inner product between two states $\tilde{f}_{\phi_1,\phi_2,\phi_3,|v|}$ and $\tilde{f}_{\phi_1,\phi_2',\phi_3',|v|'}$ is :

$$\left\langle \tilde{f}_{\phi_1,\phi_2',\phi_3',|v|'}|\tilde{f}_{\phi_1,\phi_2,\phi_3,|v|}\right\rangle \propto \int dk d\sigma_1 d\sigma_2 d\sigma_3 \delta\left(12\pi G k^2 - \sum_i {\sigma_i}^2\right)|\sigma_1| \tag{4.29}$$

$$\times\exp\left(\mathrm{i}\left(\sum_{i\in\{2,3\}}\left(\phi_i' - \phi_i\right)\sigma_i + \ln\left|\frac{v'}{v}\right|k\right)\right) \tag{4.30}$$

$$= \int dk d\sigma_2 d\sigma_3 \theta\left(12\pi G k^2 - {\sigma_2}^2 - {\sigma_3}^3\right) \tag{4.31}$$

$$\times\exp\left(\mathrm{i}\left(\sum_{i\in\{2,3\}}\left(\phi_i' - \phi_i\right)\sigma_i + \ln\left|\frac{v'}{v}\right|k\right)\right)\,. \tag{4.32}$$

Because of the Heaviside function in the integrand, we don't get Dirac distributions in the end, but kinds of projected ones : the integration is done inside a cone and not the whole spectrum.[9] This makes the conditional probabilities a lot harder to compute, even if they are perfectly well define.
What we would like to achieve in the end is to compare predictions made with different clocks. That's why we chose to work with three scalar fields : we could measure the same field ϕ_3 with respect to ϕ_1 and ϕ_2, corresponding to the probabilities $P(\phi_3|\phi_1)$ and $P(\phi_3|\phi_2)$. At the classical level, all the fields are proportional to each other, so we expect these two probabilities to code the same information. However, we don't exclude some quantum fluctuations of the clock to come up. Depending on the state of the system, the quality of the clocks should differ, and we think this could be shown by comparing the conditional probabilities. Concretely, the final goal of this study is to define semi-classical states at late times[10] and determine if the quantum nature of a real clock plays a crucial role in the picture. Obviously, we would then aim at generalize the study to LQC, for which the discrete structure of geometry can complicated our task even more. However, we believe that conceptually the issues are essentially the same in both theories, and solving them in the WdW framework would give the path to be followed in LQC.

Conclusion

In the first half of this report, we detailed some aspects of the problem of time, and presented two main strategies to solve it. The evolving constants program allows to build notions of time from timeless equations, both at the classical level and the quantum level. However we claimed that, for interpretational reasons, the use of conditional probabilities is necessary in the quantum theory. Indeed, we expect a quantum theory of gravity to feature clock fluctuations, which are not compatible with a parameter time, like that of standard quantum mechanics or evolving constants families. We only briefly described the second framework solving the problem of time. It corresponds to a radically different strategy : instead of looking for good clocks in apparently timeless theories, we simply discard time and deny its fundamental nature. Dynamics is then a set of transition amplitudes between kinematical states, labelled by possible outcomes of measurements. Time plays no role, and its flow his supposed to be an illusion.
In the second half of this document, we tried to sum up the effective work done during the internship. We focused on the evolving constants program and simple models. The first one was a non-relativistic particle, studied as a cosmological toy model, while the second was a true cosmological model of homogeneous and isotropic universes. For the non-relativistic particle, we managed to compute conditional probabilities between momentum and time,

[9] In the 2-dimensional case (only two scalar fields), Cauchy principal values would appear, since :

$$\int dxdy\theta\left(x^2-y^2\right)e^{\mathrm{i}(k_xx+k_yy)} = 2\left(\text{p.v.}\frac{\mathrm{i}}{\mathrm{k_x}-\mathrm{k_y}}+\pi\delta\left(\mathrm{k_x}-\mathrm{k_y}\right)\right)\left(\text{p.v.}\frac{\mathrm{i}}{\mathrm{k_x}+\mathrm{k_y}}+\pi\delta\left(\mathrm{k_x}+\mathrm{k_y}\right)\right) \quad (4.33)$$

$$+\left(\text{p.v.}\frac{-\mathrm{i}\sqrt{2}}{\mathrm{k_x}-\mathrm{k_y}}+\pi\delta\left(\frac{\mathrm{k_x}-\mathrm{k_y}}{\sqrt{2}}\right)\right)\left(\text{p.v.}\frac{-\mathrm{i}\sqrt{2}}{\mathrm{k_x}+\mathrm{k_y}}+\pi\delta\left(\frac{\mathrm{k_x}+\mathrm{k_y}}{\sqrt{2}}\right)\right). \quad (4.34)$$

[10] i.e. states corresponding to a mostly classical universe, as observed today.

which we then compared to standard quantum mechanics. We related the deviations from the well-known amplitudes to quantum fluctuations of the clock. These are expected to provide the theory with a fundamental mechanism of decoherence, especially in cosmological situations. As for the cosmological model studied in the last part, we unfortunately did not manage to achieve a lot. We nevertheless showed that several consistant choices of time can sometimes be made, even in the quantum theory. We can therefore easily define conditional probabilities, which we need now to explicitly compute, at least for some specific states. This would allow us to compare the different clocks.

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