The Problem of Time in Quantum Cosmology

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The problem of time in gauge theories, general relativity, quantum gravity and cosmology is addressed. We review the relational quantum evolution of dynamical systems with no time in terms of Rovelli's evolving constants, as well as the formulation of quantum mechanics with conditional probabilities and Dirac observables. This construction is applied to a model of quantum cosmology based on the Wheeler-DeWitt quantization of four dimensional gravity sourced by three scalar fields. We show that the resulting quantum theory is ill-defined and that the conditional probabilities cannot be rigorously calculated. However, we describe an appropriate framework to perform quantum cosmology since the quantum fluctuations of the clock are not neglected anymore.

I. INTRODUCTION

The search of a satisfactory quantum theory of gravity, that would represent a synthesis of general relativity and quantum mechanics, has been at the center of most research in theoretical physics for a while. Already in 1916, Einstein pointed out the necessity of a quantum theory of gravity. Many research programmes in quantum gravity have been developped throughout the years, amongst which the most popular ones are certainly superstring theory and loop quantum gravity (LQG) [25]. LQG is a non-perturbative and background independent quantization of general relativity based on canonical variables discovered by Ashtekar [4, 5], in which space-time geometry is treated quantum mechanically from the beginning. The application of LQG to symmetry reduced models of general relativity, known as loop quantum cosmology (LQC) [8], and has been widely investigated recently because it provides a simple framework to study space-time singularities (e.g. near the big-bang or inside black holes) and the cosmological implications of LQG.

All the theories of quantum gravity and quantum cosmology share the same deep philosophical questions, the most intriguing and exciting one being certainly the problem of time [17]. Time is simply one of the deepest issues that must be addressed because it is related to the foundations of quantum mechanics, the notions of space-time, causality, and evolution (not to mention many more). The notion of time we use in conventional quantum theories is deeply grounded in the concept of Newtonian physics, where there is an absolute time parameter external to any system. In generally covariant systems like general relativity, the Hamiltonian (which generates time evolution) is a constraint which vanishes identically.

Thus, there is no time parameter to describe the dynamics of the system and we fail to define evolution. The concept of an absolute Newtonian time is simply incompatible with the formulation of diffeomorphism-invariant theories. Also, we run into difficulties if we want to work out the quantum theory of a closed system like we do in quantum cosmology. It has even been argued by Barbour [7], Misner and Hawking that time cannot play an essential role in the formulation of quantum cosmology.

Most of the attempts to solve the problem of time in quantum gravity are based on identifying an internal time, *i.e.* a degree of freedom of the system (either gravitationnal or related to the matter content) that can be used to compute the evolution of the other variables. Many other promessing directions have been investigated, such as the emergence of a thermodynamical time [26], the study of unimodular gravity [29], or the use of matter degrees of freedom [30]. These approaches will not be reviewed in this work. Rovelli has introduced an elegant manner to define time evolution for systems which do not admit a general Hamiltonian structure and in which no absolute time parameter can be used [27]. This has lead to a very interesting model of quantum mechanics without time [28] in which we consider only relational aspects and correlations between physical observables called evolving constants. However, this formalism uses an external time parameter to define evolution. Gambini, Porto, Pullin and Torterolo [10] have very recently suggested a scheme that allows to reformulate timeless quantum mechanics by using Rovelli's evolving constants and abolishing any reference to the external time parameter.

In this work, we wish to review these results and show how they relate together. Ultimatelty we would like to apply this framework to quantum cosmology. In LQC, dynamics is dictated by an equation which is similar to the Wheeler-DeWitt (WDW) equation: it makes no reference to an absolute time. In actual formulations of LQC, matter is sourced by a scalar field which serves as an internal time [1] and all the predictions of the theory are relational (we can ask for the value of the volume when the scalar field has a given value). The idea is to use several scalar fields in order to study the behaviour of the theory when the quantum fluctuations of the clock are included.

This work is organized as follows. Section II is meant to show how general relativity can be seen as a gauge theory and how the notion of observability and evolution are subtle concepts in diffeomorphism-invariant theories. Section III introduces dynamical systems with no time, the definition of evolution for such systems, and gives several examples. The quantum theory of timeless systems is reviewed in section IV as well as the conditional probabilities which we attempt to use to solve the problem of time. Finally, section V presents the Wheeler-DeWitt theory at the classical and the quantum level, and we try to apply the concepts of conditional probabilites to this model of quantum cosmology.

II. GENERAL RELATIVITY AND GAUGE THEORIES

In this section we intend to show how the interpretation of general relativity as a gauge theory leads to difficulties in the definition of observables and evolution, which directly relate to the problem of time. Although a consistent Hamiltonian formulation of general relativity is not known, it is possible to interpret the theory with the more general approach used for gauge systems. Gauge systems are constrained Hamiltonian systems in which the dynamics is described with respect to an arbitrary reference frame. The observables, which are required to be independent of the choice of the local reference frame, are said to be gauge-invariant. We will see that there are difficulties in the definition of change and time in this picture.

A. Formalism

We consider as a model of general relativity a globally hyperbolic solution (M, g) to the vacuum Einstein equations, consisting of a four dimensional manifold M equipped with a Riemannian metric g. We assume that the Cauchy surfaces of M are diffeomorphic to a compact three manifold Σ, and use a diffeomorphism $\phi : \Sigma \rightarrow M$ to embed Σ in M in such a way that $S \equiv \phi(\Sigma)$ is a Cauchy surface of (M, g). We are interested in the geometry induced by g on S. Any such geometry is characterized by two symmetric tensors on S, namely q_{ab} and K_{ab}, q being the Riemannian metric induced on S, and K the extrinsic curvature, specifying how S is embedded in M. The diffeomorphism ϕ can be used to pull these tensors back on Σ, thereby defining the geometric structure of the submanifold (Σ, q, K) of (M, g).

Now if Σ comes with two symmetric tensors, q and K, we wonder under which conditions (Σ, q, K) can be viewed as being a Cauchy surface of the model (M, g). It is the case if the Gauss-Codazzi constraint equations hold:

$$R + K^2 - K^{ab}K_{ab} = 0,$$

$$\nabla^a K_{ab} - \nabla_b K = 0.$$

Here R is the Ricci scalar (*i.e.* the scalar curvature of the hypersurface Σ), and ∇ denotes the covariant derivative. Since these equations make only reference to q and K, we can regard the pair (q, K) as representing the dynamical state of the gravitational field at a given time (if and only if it satisfies the Gauss-Codazzi constraints). To reformulate general relativity as a constrained Hamiltonian system, we consider the space of Riemannian metrics on Σ, $Q \equiv \mathrm{Riem}(\Sigma)$, as the configuration space of the theory. We then construct the cotangent bundle T^*Q and endow it with the canonical symplectic structure ω. The momentum canonically conjugated to q is

$$p^{ab} \equiv \sqrt{\det q}\left(K^{ab} - Kq^{ab}\right).$$

The phase space of general relativity is given by the constraint surface $N \subset T^*Q$, where we restrict attention to the points satisfying the following first class constraints (known as scalar and vector constraints respectively):

$$\sqrt{\det q}\left(p^{ab}p_{ab} - \frac{1}{2}p^2 - R\right), \tag{2.1}$$

$$\nabla_b p^b{}_a = 0. \tag{2.2}$$

These equations, also known as the Hamiltonian and momentum constraints, are simply the Gauss-Codazzi constraints rewritten in terms of p. If we equip N with the form $\sigma = \omega|_N$ amd let $H = 0$, general relativity is the gauge theory (N, σ, H).

B. Interpretation of change and gauge invariance

In the previous subsection we have roughly seen how general relativity can be understood as a gauge theory. Consider two points of the full phase space of general relativity, which correspond to two distinct Cauchy surfaces of the same model (M, g). Since these two points are related by a gauge transformation generated by the scalar constraint (2.1), they have to be regarded as being equivalent. On the other hand, if we require the physical quantities (*i.e.* the physical observables) to be gauge-invariant, then it is not possible to use such quantities to distinguish between two Cauchy surfaces.

We can therefore ask if there is room for time and change in such gauge-invariant theories. This question has been addressed by Kuchař [19] and Unruh [31]. They wonder wether or not change can exist. Indeed, if we require the physically observable quantities to commute with all the constraints, the objects that we build are time-independant and therefore cannot describe the change in time, the evolution of the Universe. We end up with a frozen time formalism in which the time-independant quantities in general relativity are not enough to describe evolution and change. Of course this problem with the definition of evolution and change is closely related to the lack of time in general relativity. The main point of Kuchař and Unruh is the following: if we require the the physical observables to be gauge-invariant quantitites, then there is no such quantity which can take different values on two Cauchy surfaces corresponding to distinct times. Thus, general relativity can accomodate no change when viewed as a gauge theory. This apparent contradiction with the fact that we experience change in our observations enlights the fact that a theory of quantum gravity is very unlikely to be built upon such a foundation. There is a deep philosophical problem and a need for defining evolution and change in a more consistent way.

III. DYNAMICAL SYSTEMS WITHOUT TIME

Throughout this part, we are going to discuss the representation of time in mechanics, starting with the symplectic representation of Hamiltonian systems. In this picture, mechanics is seen as the theory of the evolution of a given physical system in a time variable t. However, we will see that Hamiltonian mechanics admits an elegant alternative reformulation which does not require the use of a specific time variable. Such a framework, adapted to the study of gauge theories, is provided by presymplectic geometry. We will then see how a consistent definition of observables and evolution can be made using 'evolving constants of motion'.

A. Presymplectic geometry

A Hamiltonian dynamical system is given by a triple (M, ω, H), where M is a manifold, $H : M \to \mathbb{R}$ a Hamiltonian (which is a smooth function on M), and ω a close nondegenerate two-form. The pair (M, ω), called a symplectic manifold, constitutes the phase space of some physical system, e.g. the space of particle position and conjugate momenta. If we call (q_i, p_i) the canonical coordinates on M, we have $\omega = dp_i \wedge dq_i$. The dynamics of the system is given by the Hamiltonian equations: H and ω together determine a flow on M (labeled by the time parameter t) which maps each state to the state that dynamically follows from it after a time t. This means that the motions in M are given by the integral lines of the vector field $\partial/\partial t$.

Let us now consider the manifold $S = M \times \mathbb{R}$, where $\mathbb{R}$ is the real line, in which the time t has value. Given the coordinates (q_i, p_i, t) and the so-called presymplectic form

$$\omega_s = \omega - dH(q_i, p_i, t) \wedge dt, \tag{3.1}$$

the dynamical system can now be defined on the presymplectic manifold (S, ω_s), with the equations of motion given by the integral lines of the null vector field of ω_s. Such presymplectic geometries serve as the phase spaces of gauge theories. As is clear from Eq. (3.1), the parameter t which has been promoted to a variable on the enlarged phase space S, has now a preferred role which identifies it as the time variable. It is however possible to exploit the coordinate-independence of the presymplectic space to rewrite Eq. (3.1) with a new time variable:

$$\omega_s = \omega - dH'(q'_i, p'_i, t') \wedge dt'.$$

This enlights the fact that the presymplectic formulation can accomodate different time variables. In the case of a relativistic dynamical system, any Hamiltonian formulation will single out a specific Lorentz time and therefore break the Lorentz invariance, whereas the presymplectic formulation can account for different Hamiltonian formulations, each corresponding to a different Lorentz time. In this respect, the presymplectic framework constitutes the only way to formulate the dynamics of a relativistic system without breaking its Lorentz invariance.

In such a presymplectic formulation, the observables of the system are the scalar functions on S that are constants along the trajectories, and the states of the system correpond to the orbits of the presymplectic form ω_s. Since these two definitions make no reference to a preferred definition of time, the presymplectic framework outlined above can also decribe systems in which there is no Hamiltonian time t at all. Such systems, which do admit a presymplectic but no symplectic (*i.e.* Hamiltonian) formulation, are common in physics. A typical dynamical system without time is of course general relativity, a generally covariant theory in which the Hamiltonian is a constraint and vanishes identically.

B. Description of evolution

Let us consider a Hamiltonian constrained theory with a first class constraint algebra. The Lie bracket between two constraints is given by

$$\{\mathcal{C}_i, \mathcal{C}_j\} = f_{ij}^k \mathcal{C}_k,$$

f_{ij}^k being the structure constants. Constraints generate gauge orbits in the phase space and the observables of the system are the scalar functions D that are constant along these orbits. Such a definition of an observable (in the sense of Dirac) is equivalent to the requirement that the observable has vanishing Poisson brackets with the constraints. Hence, we have

$$\{D, \mathcal{C}_j\} = 0 \;\; \forall\, j.$$

In relativistic theories of gravity like general relativity there is a time reparametrization invariance of the action which implies that the Hamiltonian H_s (often called super-Hamiltonian) vanishes. The generator of the evolution also generates gauge transformations.

Since we require the Dirac observables to be gauge-invariant quantities, we have

$$\{D, H_s\} = 0, \tag{3.2}$$

and the observables are constants of motion. At the quantum level, the operator $\hat{H}_s$ annihilates physical states and commutes with the observables:

$$[\hat{D}, \hat{H}_s] = 0. \tag{3.3}$$

The last two equations seem to prohibit any reparametrization-invariant classical and quantum evolution. In other words, how can we describe evolution if the physically relevant quantities are constants of motion?

The proposal of Rovelli is to define evolution using families of observables labeled by a time parameter τ [27]. We can choose a time function on the extended phase space, which is such that its level surfaces are oblique to the gauge orbits. To explain how this is possible, let us consider a presymplectic system (with a constraint $\mathcal{C}$) on a $2n$ dimensional phase space $(q_1, \ldots, p_n)$, and single out a variable, say q_1, to play the role of a clock variable. The solution to the apparent problem of evolution is that we can build observables that satisfy Eqs. (3.2) and (3.3), and represent evolution in the variable q_1. Let us construct the observable that represents the evolution of the variable q_i (for $i = 2, \ldots, n$) as a function of the clock variable q_1. We call $Q_{[q_i,q_1]}(\tau)$ the one-parameter family of observables that gives the value of q_i for each value τ of the clock variable q_1. For $Q_{[q_i,q_1]}(\tau)$ to be an observable, it has to be constant along the orbits for every τ. At each point of the constraint surface, we require it to be equal to the function $q_i(q_1)$. This definition of the observable is equivalent to the following equations:

$$\{Q_{[q_i,q_1]}(\tau), \mathcal{C}\} = 0,$$

$$Q_{[q_i,q_1]}(q_1) = q_i.$$

The first equation ensures that $Q_{[q_i,q_1]}(\tau)$ is constant along the trajectories and the second one determines its value on any trajectory. Any observable defined in such a way is called an evolving constant of motion. Kuchař interprets this construction by saying that change in a quantity that is not gauge-invariant can be observed indirectly by observing gauge-independant quantities and looking at their relations. The evolving constants meet the challenge of defining time and change in the context of general relativity [2].

We shall now give some explicit examples of the construction of observables as evolving constants. In particular we will see that for certain systems, different choices of the time function can be made, and some modifications are required at the quantum level to make the definition of evolution fully satisfactory.

C. Exemples of parametrized systems

As noticed by Dirac [9] and Kuchař [20], time reparametrization invariance can be obtained for any classical theory by elevating time to the status of a canonical variable. By doing so, we end up working in the $(2n+2)$ dimensional extended phase space, which contains time and its conjugate momentum, (q_0, p_0), as well as the $2n$ spatial phase space variables,

$(q_1, \dots, p_n) := (q_i, p_i)$. The collection of extended phase space variables is here denoted (q, p). We consider a classical action for a point particle in a parametrized form

$$S = \int d\lambda \Big(p\dot{q} - N(\lambda) H_s(q, p) \Big),$$

where $H_s(q, p)$ is the super-Hamiltonian (a first class constraint), $N(\lambda)$ the lapse function, and the dot indicates differentiation with respect to the affine parameter λ. The super-Hamiltonian is given by

$$H_s(q, p) = p_0 + H(q_i, p_i),$$

where $H(q_i, p_i)$ is the classical time-independant Hamiltonian of a non-parametrized free particle. The variation of the lapse in the action gives the constraint

$$H_s(q, p) \approx 0,$$

where the $\approx$ sign means 'to vanish as a constraint' in the sense of Dirac. It can be set to zero only after all the Poisson brackets have been evaluated. This first class constraint then defines the constraint surface in the extended phase space, and generates classical trajectories according to the Heisenberg equations of motion

$$\dot{q} = N(\lambda)\{q, H_s\}, \quad \text{and} \quad \dot{p} = N(\lambda)\{p, H_s\}.$$

The set of classical trajectories defined by these equations constitute what we call the reduced phase space, and the observables are precisely the scalar functions that are constant along these classical trajectories. Since the super-Hamiltonian is viewed as a constraint that vanishes, we expect the parametrized system to display the same conceptual problems as general relativity. In particular, there can be no gauge-invariant quantities that distinguish between two points lying on the same dynamical trajectory.

In the case of a single non-relativistic free particle, we have the extended phase space (q_0, q_1, p_0, p_1), the super-Hamiltonian

$$H_s = p_0 + \frac{p_1^2}{2m},$$

and the standard Poisson brackets

$$\{q_1, p_1\} = \{q_0, p_0\} = 1.$$

Note that q_0 or q_1 are not observables in the sense of Dirac. It is important to notice that the position q_1 of the particle does not commute with the constraint given by H_s, and hence is not a gauge-invariant quantity. Thus, the parametrized system seems to describe a situation in which there is no change (and no motion) at all. This is an apparent paradox in the sense that the parametrized system is describing the original Hamiltonian system in which there is a Newtonian time and a well-defined evolution. Of course, we could deparametrize the system using t as the absolute Newtonian time and recover the standard dynamics, but here we wish to circumvent the use of an absolute time and describe change in the parametrized formulation.

Following what has been discussed in the previous subsection, we can build a set of Dirac observables for this system using the evolving constants of motion. A natural choice for a clock variable is q_0, and we can build an observable giving the position q_1 of the particle when the time q_0 takes the value τ. Such an observable is given by

$$Q_{[q_1,q_0]}(\tau) = q_1 + \frac{p_1}{m}(\tau - q_0),$$

and is parametrized by τ. The observable Q gives the value of q_1 at the point on the gauge orbit where q_0 takes the value τ (it has a relational interpretation). Indeed, we have

$$\{Q_{[q_1,q_0]}, H_s\} = 0, \quad \text{and} \quad Q_{[q_1,q_0]}(q_0) = q_1.$$

Note that we have to assume that q_0 is physically observable even though it is not a Dirac observable.

Let us now consider the example of two parametrized non-relativistic free particles. The associated constraint is

$$H_s = p_0 + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2},$$

and the positions of the two particles are given by the following observables:

$$Q_{[q_1,q_0]}(\tau) = q_1 + \frac{p_1}{m_1}(\tau - q_0),$$
$$Q_{[q_2,q_0]}(\tau) = q_2 + \frac{p_2}{m_2}(\tau - q_0).$$

Now one could also use the position q_2 of the second particle as a clock variable, and build an observable giving the position q_1 of the first particle at the moment at which the second particle has the position τ'. This observable is simply

$$Q_{[q_1,q_2]}(\tau') = q_1 + \frac{p_1}{m_1}\frac{m_2}{p_2}(\tau' - q_2).$$

We see that different choices of time functions are possible, leading to similar predictions at the classical level but to differences when we turn to quantization. Several problems arizing in such a framework have been emphasised by Kuchař [18, 19]. In particular, when we use the position of a particle as the time function, there is a momentum in the denominator of the evolving constant, which leads to an ill-defined quantum operator. In the next section we are going to review the quantization of systems without time and discuss the appropriate framework to use quantum evolving constants.

IV. TIME AND THE QUANTUM

In this section we wish to show how the definition of time and evolution provided by the use of evolving constants can be transposed to the quantum theory. The quantization of constrained systems is well-known since the work of Dirac [9]. Consider a dynamical system (M, ω, H), that we wish to quantize into a quantum system $(\mathcal{H}, A_i, H)$ given by a Hilbert space $\mathcal{H}$ equipped with a Hamiltonian operator $\hat{H}$ and a representation of the algebra of classical observables A_i (that is found by first ignoring the constraints). We chose the constrained system to be described by the submanifold $(N, \sigma \equiv \omega|_N)$ of the symplectic system (M, ω), determined by the first class constraint $\mathcal{C}$. To quantize this system, one needs a set of coordinates on the manifold M, say the canonical coordinates (p, q), and a vector space V carrying a representation of their Poisson algebra as linear operators. One then defines the operators $\hat{q}$ and $\hat{p}$ satisfying the commutation relation $[\hat{p}, \hat{q}] = i\hbar$, and builds the quantum constraint $\hat{\mathcal{C}}$. The physical states are annihilated by the constraint operator, which ensures that the quantum states are gauge-invariant and enables us to build the space of physical states V_{phys}. Then one looks for an inner product that makes V_{phys} into a Hilbert space, and for a Hamiltonian operator $\hat{H}$ that gives the quantum dynamics according to Schrödinger's equation.

Quantum mechanics without time has been investigated by Rovelli using the formalism of evolving constants [27, 28]. The standard formulation of quantum mechanics requires the use of an absolute Newtonian time which is used to define the evolution of a system. If we have a classical observable $Q(t)$ which evolves according to Hamilton's equation of motion

$$\dot{Q}(t) \equiv \frac{\partial Q(t)}{\partial t} = \{Q(t), H\}, \tag{4.1}$$

with the Hamiltonian H, we expect at the quantum level that the Heisenberg observable $\hat{Q}$ obeys the following evolution equation:

$$\hat{Q}(t + t') = \exp\left(i\hbar t' \hat{H}\right) \hat{Q}(t) \exp\left(-i\hbar t' \hat{H}\right), \tag{4.2}$$

where $\hat{H}$ is the quantum Hamiltonian operator. The quantum equivalent of Eq. (4.1) is the Schrödinger equation

$$i\hbar \frac{\partial \hat{Q}(t)}{\partial t} = [\hat{Q}(t), \hat{H}] \tag{4.3}$$

which carries an absolute Newtonian time structure and is here written in the Heisenberg picture (*i.e.* with time-dependant operators). We now come to the main point of Rovelli's work. We have seen that for classical systems without time, the Hamilton equation (4.1) does not hold. We therefore expect that in the quantum analog of such a timeless system, the Schrödinger equation (4.3) will not hold either. We can build an evolving constant $Q(\tau)$ with a corresponding quantum operator $\hat{Q}(\tau)$ that does not satisfy Eqs. (4.2) and (4.3). In fact this does not affect the Heisenberg picture in any way. The Heisenberg states are the quantum analogs of the presymplectic states and quantum systems without time can still be defined even if no Schrödinger picture holds. Since general relativity is a system with no time structure, we search for a formulation of quantum gravity which is built on this framework of timeless quantum mechanics. The quantum description that has been built in term of evolving constants is purely relational, we use certain dymanical variables of the system as clocks and look at the evolution of a physical quantity described by an evolving constant.

A. Quantum evolving observables

Parametrized non-relativistic systems do admit a natural time structure. However, the use of evolving constants allows us to use different clock variables, and we wonder to what extend the resulting quantum theories are equivalent. It has been shown that the use of q_0 as the clock variable leads to predictions that coincide with those of usual non-relativistic quantum mechanics [16].

Let us go back to the example of a non-relativistic free particle that has been introduced in section III. We recall that the extended phase space is (q_0, q_1, p_0, p_1), and that the constraint is given by the super-Hamiltonian

$$H_s = p_0 + \frac{p_1^2}{2m}.$$

To pass to quantum mechanics, let us consider the linear space of wave functions Ψ and represent the canonical coordinates on the phase space manifold by operators in the usual way: the position acts as a multiplication operator and its conjugate momentum as $p_\mu = -i\partial/\partial q_\mu$. Physical states are annihilated by the operator form of the constraint. Let us now introduce the auxilliary Hilbert space $\mathcal{H}_{\text{aux}} = L^2(\mathbb{R}^2, dp_0\, dp_1)$ on which the quantum operators corresponding to the classical observables p_1 and $q_1 - q_0 p_1/m$ are self-adjoint. On this auxilliary Hilbert space, the quantum operator $\hat{Q}$ corresponding to the evolving constant $Q_{[q_1,q_0]}(\tau)$ is also self-adjoint. Here we do not wish to review the whole process of refined algebraic quantization, we rather refer the reader to [3, 21–23]. Let us just mention that the physical states can be considered as distributions $f(p_0, p_1)$ acting on a dense subspace $S_L \subset \mathcal{H}_{\text{aux}}$. We can show that if we consider solutions of the form

$$\Psi(p_0, p_1) = \delta\left(p_0 + \frac{p_1^2}{2m}\right) f(p_0, p_1),$$

we can introduce the physical inner product

$$\langle \Psi_1(p_0, p_1)|\Psi_2(p_0, p_1)\rangle_{\text{phys}} = \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_0\, \delta\left(p_0 + \frac{p_1^2}{2m}\right) \bar{f}_1(p_0, p_1) f_2(p_0, p_1),$$

and obtain the physical Hilbert space $\mathcal{H}_{\text{phys}}$ by considering a completion of S_L. It has been shown that the choice of clock variable q_0 leads to the standard form of the quantum free particle in the Heisenberg representation [14]. In particular, we recover the usual transition amplitude.

The parametrized formulation of the non-relativistic free particle dynamics permits more choices of time functions. However, it has been stressed by Kuchař [20] that any other choice leads to evolving Dirac observables that cannot be promoted to self-adjoint operators. For example, one could take the position q_1 of the particle as the clock variable. Appart from those with zero momentum, all classical trajectories intersect a surface $\tau = q_1$ once and only once, which suggests that the position might be a good clock variable. One could then write down the operators $\hat{q}_0$, $\hat{q}_1$, $\hat{p}_0$ and $\hat{p}_1$ corresponding to the phase space coordinates when $\tau = q_1$ and build the evolving observable defined by

$$Q_{[q_0,q_1]}(\tau) = \frac{m}{p_1}(\tau - q_1), \qquad \text{and} \qquad Q_{[q_0,q_1]}(q_1) = q_0.$$

We see a problem arising because of the factor $1/\hat{p}_1$ in the denominator. Since zero is in the continuous part of the spectrum of his operator, authors have argued that we cannot build a self-adjoint operator (this might not be a problem is we consider the fact that any measurable function of a self-adjoint operator is a self-adjoint operator).

Other criticisms focus on the issue of the parameter τ used to define time evolution of the evolving constants. It is a real parameter which is not represented by any quantum operator and does not belong to any physical Hilbert space. When we use the clock variable q_0, we wonder what sense to give to the condition $\tau = q_0$ if q_0 is not defined in the physical Hilbert space, *i.e.* if

$$q_0|\Psi\rangle_{\text{phys}} \notin \mathcal{H}_{\text{phys}}.$$

For the non-relativistic particle, this is not a problem because we can simply consider τ as being the time measured by an external clock. However if we want to consider generally covariant systems like general relativity, the clock has to be associated to certain physical sub-system with dynamical variables that will not be well defined in $\mathcal{H}_{\text{phys}}$. Summarizing, evolving observables are measurable quantities but they depend on an external parameter whose observation is not described by the theory.

An alternative has been suggested by Page and Wootters [24], who used conditional probabilities to describe evolution. The idea is to promote all the variables of the system to quantum operators and then choose one variable to play the role of a clock. Conditional probabilities can then be computed to find out the value taken by the other variables when the clock has a given value. Kuchař has shown [19] that we cannot recover the correct propagator in this framework because, here again, the evolution variable cannot be defined in the physical space of states.

B. Conditional probabilities in terms of evolving Dirac observables

We present here the framework developped by Gambini, Porto, Pullin and Torterolo [10] to abolish all reference to external parameters and to define correlations between Dirac observables of the theory using evolving constants. We shall assume that the evolving constants are measurable quantities, provided they can be promoted to well defined self-adjoint operators in the physical Hilbert space. To introduce a fully quantum relational description of evolution, we need to get rid of the dependence on external parameters such as the clock variable. From a physical point of view, any observable in the theory should be described by a quantum operator in the space of physical states, and that should not be different for time, which is ultimately measured by physical clock obeying the laws of quantum mechanics.

The first step is to quantize the evolving constants and to choose one of the variables, say $T(t)$, to be a quantum clock. If we choose our quantum observable to be $Q(t)$, we can compute the conditional probability [12]

$$\mathcal{P}\left(Q \in [Q_0 - \Delta Q, Q_0 + \Delta Q] | T \in [T_0 - \Delta T, T_0 + \Delta T]\right) \equiv \lim_{\tau\to\infty} \frac{\int_{-\tau}^{+\tau} dt\, \mathrm{Tr}\left[P_{Q_0}(t) P_{T_0}(t) \rho P_{T_0}(t)\right]}{\int_{-\tau}^{+\tau} dt\, \mathrm{Tr}\left[P_{T_0}(t)\rho\right]},$$

where $P_{Q_0}(t)$ is the projector on the eigenspace associated with eigenvalue Q_0 at time t, and similarly for $P_{T_0}(t)$. Here we integrate in the external time parameter t, which is treated as an ideal unobservable quantity. This conditional probability is gauge-invariant since the density matrix ρ is assumed to be annihilated by the constraint. It has been shown that the conditional probabilities yield the correct propagator for a simple system consisting of two particles. This framework is the only one that allows to access measurable quantities in systems in which there is no absolute and ideal time.

Moreover, this framework allows to account for real clocks which are subjected to quantum fluctuations [11, 13]. If we use such real physical clocks, there is a fundamental decoherence and a loss of unitarity due to the quantum fluctuations of the clock. The unitary evolution of the evolving constants in the ideal time t is unobservable.

Now, we wish to apply these techniques to the problem of time in quantum cosmology. We shall therefore present the Wheeler-DeWitt theory of homogeneous and isotropic cosmological models and try to compute conditional probabilities using different clock variables.

V. WHEELER-DEWITT THEORY

In this section we introduce the WDW quantization in order to illustrate some of the basic ideas relative to the quantization scheme in LQC. In particular, we consider a model in which matter is in the form of three massless scalar fields, and investigate the effect of the parametrization and the choice of the clock variable. To this end, we build a set a Dirac observables, semiclassical states, and compute conditional probabilities for the observables. The quantization, carried out in the connection-triad variables, can be seen as Schrödinger-like quantization of the ADM Hamiltonian.

A. Classical level

For symplicity we restrict ourselves to the flat ($k = 0$), homogeneous, and isotropic model. The action we wish to quantize is the action of four dimensional gravity to which we add the action for the three free scalar fields. It is given by [32]

$$S = \int dt\, \frac{3}{\kappa\gamma} p\dot{c} + p_{\phi_1}\dot{\phi}_1 + p_{\phi_2}\dot{\phi}_2 + p_{\phi_3}\dot{\phi}_3 - N(t)\left[-\frac{3}{\kappa\gamma^2}\sqrt{|p|}c^2 + \frac{p_{\phi_1}^2 + p_{\phi_2}^2 + p_{\phi_3}^2}{2|p|^{3/2}}\right].$$

Let us solve the classical equations of motion. If we write $\Pi_\phi^2 := p_{\phi_1}^2 + p_{\phi_2}^2 + p_{\phi_3}^2$, the vanishing of the Hamiltonian constraint gives

$$c = \pm\sqrt{\frac{\kappa\gamma^2}{6}}\frac{\Pi_\phi}{|p|}.$$

Since the scalar fields have no potentials, the scalar fields momenta p_{ϕ_i} (with i=1, 2, 3) are constants of motion, as shown by Hamilton's equations

$$\dot{p}_{\phi_i} = \{p_{\phi_i}, N\mathcal{C}\} = 0. \tag{5.1}$$

The Hamiltonian equations for p read

$$\begin{aligned} \dot{p} = \{p, N\mathcal{C}\} &= -\frac{\kappa\gamma N}{3}\frac{\partial\mathcal{C}}{\partial c} \\ &= \frac{2N}{\gamma}\sqrt{|p|}c \\ &= \pm N\sqrt{\frac{2\kappa}{3}}\frac{\Pi_\phi}{\sqrt{|p|}}. \end{aligned} \tag{5.2}$$

From now on, it is convenient to make a gauge choice in which the lapse function $N(t)$ is equal to $|p|^{3/2}$. We can then integrate Eq. (5.2) to obtain

$$p(t) = \pm p_0 \exp\left(\pm\sqrt{\frac{2\kappa}{3}}\Pi_\phi(t-t_0)\right). \tag{5.3}$$

With this choice of gauge, the solution is not generically singular at $t = t_0$. It is important to notice that the triad p is a monotonic function of time for the four solutions described. These solutions correspond to two different orientations of the triad (plus or minus outside the exponential) and either an expanding or a contracting universe (plus or minus respectively inside the exponential). Hamilton's equations for the scalar fields are

$$\dot{\phi}_i = \{\phi_i, N\mathcal{C}\} = p_{\phi_i},$$

which can be integrated to give

$$\phi_i(t) = p_{\phi_i}(t-t_0) + \phi_i^0. \tag{5.4}$$

Now, using Eqs. (5.3) and (5.4), the solutions can be deparametrized thereby removing any reference to the unphysical time parameter t. We obtain the following expressions:

$$\begin{aligned} p(\phi_i) &= \pm p_0 \exp\left(\pm\sqrt{\frac{2\kappa}{3}}\frac{\Pi_\phi}{p_{\phi_i}}(\phi_i-\phi_i^0)\right), \\ \phi_i(p) &= \pm\sqrt{\frac{3}{2\kappa}}\frac{p_{\phi_i}}{\Pi_\phi}\ln\left(\frac{p}{p_0}\right) + \phi_i^0. \end{aligned}$$

For two scalar fields ϕ_i and ϕ_k, we can also construct the function

$$\phi_i(\phi_k) = \frac{p_{\phi_i}}{p_{\phi_k}}(\phi_k - \phi_k^0) + \phi_i^0.$$

B. Wheeler-DeWitt quantization

We now proceed to the Dirac quantization of the WDW theory. The first step is to find a quantum representation of the algebra of elementary variables, and then construct the kinematical Hilbert space. In the simple case where we consider only one scalar field ϕ, the kinematical Hilbert space is $\mathcal{H}_{\rm kin}^{\rm wdw} = L^2(\mathbb{R}^2, d\mu(p)\,d\phi)$, where $d\mu(p)$ is a measure on p chosen so that the constraint operator is self-adjoint with respect to the kinematical inner product. The WDW equation can be cast in the form of a Schrödinger equation evolving with respect to the internal time variable ϕ and the physical sector is constructed by choosing wavefunctions that are solutions to the constraint equations.

Let us now derive the WDW equation with threee scalar fields. As in usual quantum mechanics, we choose operators acting in the following way:

$$\begin{aligned} \hat{p}\,\Psi(p,\phi_1,\phi_2,\phi_3) &= p\,\Psi(p,\phi_1,\phi_2,\phi_3), \\ \hat{\phi}_i\,\Psi(p,\phi_1,\phi_2,\phi_3) &= \phi_i\,\Psi(p,\phi_1,\phi_2,\phi_3), \end{aligned}$$

$$\hat{c}\,\Psi(p,\phi_1,\phi_2,\phi_3) = \frac{i\kappa\gamma\hbar}{3}\frac{\partial}{\partial p}\Psi(p,\phi_1,\phi_2,\phi_3),$$
$$\hat{p}_{\phi_i}\,\Psi(p,\phi_1,\phi_2,\phi_3) = -i\hbar\frac{\partial}{\partial\phi_i}\Psi(p,\phi_1,\phi_2,\phi_3).$$

The Hamiltonian constraint operator to quantize is given by

$$\hat{\mathcal{C}} = -\frac{3}{\kappa\gamma^2}(\hat{p}\hat{c})(\hat{p}\hat{c}) + \frac{1}{2}\left(\hat{p}^2_{\phi_1} + \hat{p}^2_{\phi_2} + \hat{p}^2_{\phi_3}\right),$$

where the choice of factor ordering in the first term will become clear in what follows. Requiering the physical wavefunctions to be annihilated by the self-adjoint constraint operator $\hat{\mathcal{C}}$, we obtain the WDW equation

$$\frac{\kappa\hbar^2}{3}p\frac{\partial}{\partial p}\left(p\frac{\partial}{\partial p}\Psi(p,\phi_1,\phi_2,\phi_3)\right) - \frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2} + \frac{\partial^2}{\partial\phi_3^2}\right)\Psi(p,\phi_1,\phi_2,\phi_3) = 0. \tag{5.5}$$

The constraint operator commutes with the parity operator Π which flips the triad orientation, acting as $\Pi\Psi(p) = \Psi(-p)$ [1]. It therefore preserves the space of wave function under consideration, namely, the eigenspace of Π with eigenvalue $+1$.

At this point, it is possible to choose a scalar field that will play the role of time and solve the WDW equation. Choosing the scalar field ϕ_1 to play the role of emergent time in our theory, we can rewrite Eq. (5.5) as

$$\begin{aligned}\frac{\partial^2}{\partial\phi_1^2}\Psi(p,\phi_1,\phi_2,\phi_3) &= \frac{2\kappa}{3}p\frac{\partial}{\partial p}\left(p\frac{\partial}{\partial p}\Psi(p,\phi_1,\phi_2,\phi_3)\right) - \left(\frac{\partial^2}{\partial\phi_2^2} + \frac{\partial^2}{\partial\phi_3^2}\right)\Psi(p,\phi_1,\phi_2,\phi_3)\\ &:= -\Theta\Psi(p,\phi_1,\phi_2,\phi_3).\end{aligned} \tag{5.6}$$

The operator Θ is self-adjoint on the Hilbert space $L^2_s(\mathbb{R}^3, d\mu(p)\,d\phi_2\,d\phi_3)$, the measure on p being $d\mu(p) := |p|^{-1}dp$, and the subscript s denoting restriction to the symmetric eigenspace of Π (*i.e.* to functions which are symmetric under p-reflexion). Θ is the difference between two positive definite operators. The first one, $-(2\kappa/3)p\partial_p p\partial_p$, has eigenvalues $\omega^2 \geq 0$, and eigenfunctions $e_k(p)$ that are 2-fold degenerate and can be labeled by a real number k:

$$e_k(p) := \frac{1}{\sqrt{2\pi}}\exp\left(ik\ln|p|\right),$$

where $\omega = \sqrt{2\kappa/3}|k|$. The second one, $-\left(\partial^2_{\phi_2} + \partial^2_{\phi_3}\right)$, has eigenfunctions which are given by

$$e_{\sigma_j}(\phi_j) := \frac{1}{\sqrt{2\pi}}\exp\left(i\sigma_j\phi_j\right),$$

where $j = 2, 3$ and the eigenvalues σ_j are real numbers. A solution to (5.6) with initial data in the Schwartz space of rapidly decreasing functions, can be written as

$$\Psi(p,\phi_1,\phi_2,\phi_3) = \int dk \int d\sigma_2 \int d\sigma_3\,\theta(\omega^2 - \sigma_2^2 - \sigma_3^2)\tilde{\Psi}(k,\sigma_2,\sigma_3)e_k(p)e_{\sigma_2}(\phi_2)e_{\sigma_3}(\phi_3)e^{i\Omega\phi_1}, \tag{5.7}$$

where $\theta(\omega^2-\sigma_2^2-\sigma_3^2)$ is a step function which ensures that the condition $\Omega^2 = \omega^2-\sigma_2^2-\sigma_3^2 \geq 0$ is respected, and $\tilde{\Psi}(k, \sigma_2, \sigma_3)$ is also in the Schwartz space. Here we have restricted ourself to the superselected sector of positive frequency. We can work in the subspace $\mathcal{H}^{\rm wdw}_{{\rm kin}\,+}$ of $\mathcal{H}^{\rm wdw}_{\rm kin}$, where the WDW operator Θ is positive definite and self-adjoint, and use the spectral decomposition of Θ on $\mathcal{H}^{\rm wdw}_{{\rm kin}\,+}$ to construct the positive definite and self-adjoint operator $\sqrt{\Theta}$. The evolution equation for the wavefunction $\Psi(p, \phi_1, \phi_2, \phi_3)$ can be obtained by taking the square-root of (5.6), which gives a first order Schrödinger equation with a non-standard Hamiltonian $\sqrt{\Theta}$:

$$\pm i\frac{\partial}{\partial\phi_1}\Psi(p, \phi_1, \phi_2, \phi_3) = \sqrt{\Theta}\Psi(p, \phi_1, \phi_2, \phi_3).$$

Since zero is in the continuous part of the spectrum of the WDW operator Θ, solutions (5.7) to the WDW equation are not normalizable in $\mathcal{H}^{\rm wdw}_{\rm kin}$. Therefore, we need to endow the space of these physical states with a Hilbert space structure. To do so, we need to find a complete set of Dirac observables and a suitable inner product by requiering that they be self-adjoint. As can be seen from Eq. (5.1) in the classical theory, the variables p_{ϕ_i} are constants of motion and hence Dirac observables. Although p is not a constant of motion, on each dynamical trajectory $p(\phi_1)$ is a monotonic function of ϕ_1, whence $p|_{\phi_1=\phi_1^0}$ is a Dirac observable for any fixed value ϕ_1^0 of the parameter ϕ_1. Since we are only considering states which are symmetric under p-reflexion, we can work with the Dirac observable $|p|_{\phi_1^0}$. Similarly, while ϕ_2 and ϕ_3 are not constants of motion, $\phi_2|_{\phi_1^0}$ and $\phi_3|_{\phi_1^0}$ are Dirac observables for any fixed ϕ_1^0. We can now use the operator $\sqrt{\Theta}$ to evolve states from a fixed ϕ_1^0 to any value ϕ_1 and compute the action of the Dirac observables on the wavefunctions:

$$\begin{aligned}
|\hat{p}|_{\phi_1^0}\,\Psi(p, \phi_1, \phi_2, \phi_3) &= \exp\left(i\sqrt{\Theta}(\phi_1-\phi_1^0)\right)|p|\,\Psi(p, \phi_1^0, \phi_2, \phi_3),\\
\hat{p}_{\phi_1}\,\Psi(p, \phi_1, \phi_2, \phi_3) &= \hbar\sqrt{\Theta}\,\Psi(p, \phi_1^0, \phi_2, \phi_3),\\
\hat{p}_{\phi_2}\,\Psi(p, \phi_1, \phi_2, \phi_3) &= -i\hbar\exp\left(i\sqrt{\Theta}(\phi_1-\phi_1^0)\right)\frac{\partial}{\partial\phi_2}\,\Psi(p, \phi_1^0, \phi_2, \phi_3),\\
\hat{p}_{\phi_3}\,\Psi(p, \phi_1, \phi_2, \phi_3) &= -i\hbar\exp\left(i\sqrt{\Theta}(\phi_1-\phi_1^0)\right)\frac{\partial}{\partial\phi_3}\,\Psi(p, \phi_1^0, \phi_2, \phi_3),\\
\hat{\phi}_2|_{\phi_1^0}\,\Psi(p, \phi_1, \phi_2, \phi_3) &= \exp\left(i\sqrt{\Theta}(\phi_1-\phi_1^0)\right)\phi_2\,\Psi(p, \phi_1^0, \phi_2, \phi_3),\\
\hat{\phi}_3|_{\phi_1^0}\,\Psi(p, \phi_1, \phi_2, \phi_3) &= \exp\left(i\sqrt{\Theta}(\phi_1-\phi_1^0)\right)\phi_3\,\Psi(p, \phi_1^0, \phi_2, \phi_3).
\end{aligned}$$

The inner product which makes these operators self-adjoint is

$$\langle\Psi_1|\Psi_2\rangle^{\phi_1=\phi_1^0}_{\rm phys} = \int|p|^{-1}dp\int d\phi_2\int d\phi_3\,\bar{\Psi}_1(p, \phi_1, \phi_2, \phi_3)\Psi_2(p, \phi_1, \phi_2, \phi_3). \tag{5.8}$$

The physical Hilbert space can be decomposed as $\mathcal{H}^{\rm wdw}_{\rm phys} = \mathcal{H}^{\rm wdw}_{{\rm phys}\,+}\oplus\mathcal{H}^{\rm wdw}_{{\rm phys}\,-}$, where the subscripts plus and minus refer to the subspaces where $\omega^2-\sigma_2^2-\sigma_3^2 \geq 0$ and $\omega^2-\sigma_2^2-\sigma_3^2 < 0$ respectively. With respect to the inner product (5.8), we have the property $\mathcal{H}^{\rm wdw}_{{\rm phys}\,+}\perp\mathcal{H}^{\rm wdw}_{{\rm phys}\,-}$. Indeed, for two solutions $\Psi_1 \in \mathcal{H}^{\rm wdw}_{{\rm phys}\,+}$ and $\Psi_2 \in \mathcal{H}^{\rm wdw}_{{\rm phys}\,-}$, we can verify that $\langle\Psi_1|\Psi_2\rangle = 0$. The action of the operator $|\hat{p}|$ on the state $\Psi(p, \phi_1, \phi_2, \phi_3)$, after projection on the subspace $\mathcal{H}^{\rm wdw}_{\rm phys\ +}$, corresponds to

$$\begin{aligned}
&\int dk \int d\sigma_2 \int d\sigma_3\, \theta(\omega^2-\sigma_2^2-\sigma_3^2)\tilde{\Psi}(k,\sigma_2,\sigma_3)|p|e_k(p)e_{\sigma_2}(\phi_2)e_{\sigma_3}(\phi_3)e^{i\Omega\phi_1} \\
=& \int dk \int d\sigma_2 \int d\sigma_3\, \theta(\omega^2-\sigma_2^2-\sigma_3^2)\tilde{\Psi}(k,\sigma_2,\sigma_3)e^{-i\partial/\partial k}e_k(p)e_{\sigma_2}(\phi_2)e_{\sigma_3}(\phi_3)e^{i\Omega\phi_1} \\
=& \int dk \int d\sigma_2 \int d\sigma_3\, \theta(\omega^2-\sigma_2^2-\sigma_3^2)e^{i\partial/\partial k}\tilde{\Psi}(k,\sigma_2,\sigma_3)e_k(p)e_{\sigma_2}(\phi_2)e_{\sigma_3}(\phi_3)e^{i\Omega\phi_1}.
\end{aligned}$$

Now if we consider an eigenstate

$$|p,\phi_2,\phi_3\rangle = \sqrt{|p|}\int dk \int d\sigma_2 \int d\sigma_3\, \theta(\omega^2-\sigma_2^2-\sigma_3^2)e_k\left(\frac{\tilde{p}}{p}\right)e_{\sigma_2}(\tilde{\phi}_2-\phi_2)e_{\sigma_3}(\tilde{\phi}_3-\phi_3),$$

where the tilde denotes the eigenvalues, the scalar product between two eigenstates is given by

$$\begin{aligned}
\langle p,\phi_2,\phi_3|p',\phi_2',\phi_3'\rangle =& \int d\tilde{p}\int d\tilde{\phi}_2\int d\tilde{\phi}_3\int dk\int d\sigma_2\int d\sigma_3\int dk'\int d\sigma_2'\int d\sigma_3' \\
&\times \theta(\omega^2-\sigma_2^2-\sigma_3^2)\theta(\omega'^2-\sigma_2'^2-\sigma_3'^2) \\
&\times \bar{e_k}\left(\frac{\tilde{p}}{p}\right)e_{k'}\left(\frac{\tilde{p}}{p'}\right)\bar{e}_{\sigma_2}(\tilde{\phi}_2-\phi_2)e_{\sigma_2'}(\tilde{\phi}_2-\phi_2')\bar{e}_{\sigma_3}(\tilde{\phi}_3-\phi_3)e_{\sigma_3'}(\tilde{\phi}_3-\phi_3') \\
=& \int dk\int d\sigma_2\int d\sigma_3\, \theta(\omega^2-\sigma_2^2-\sigma_3^2)e_k\left(\frac{p}{p'}\right)e_{\sigma_2}(\phi_2-\phi_2')e_{\sigma_3}(\phi_3-\phi_3').
\end{aligned}$$

At this point we notice that the eigenstates are not orthonormal. This relates to the fact that the WDW operator, as it has been constructed, is self-adjoint only under a certain condition (which is a restriction to the positive part of the spectrum). Therefore, if the operator is not self-adjoint, eigenstates corresponding to different eigenvalues are not generally orthonormal [15]. We would find orthonormal states if the step function θ was not present in this scalar product. However, we could continue to work as if the eigenstates were orthogonal, with an inner product given by

$$\langle p,\phi_2,\phi_3|p',\phi_2',\phi_3'\rangle = f(p,\phi_2,\phi_3)\delta(p-p')\delta(\phi_2-\phi_2')\delta(\phi_3-\phi_3').$$

Indeed, it has been suggested that the orthonormality relation still holds if we restrict ourself to the part of the Hilbert space where the WDW operator is self-adjoint and positive definite [6].

C. Conditional probabilities

Now let us choose the evolving observable $\phi_2(\phi_1)$ as our clock, the unobservable deparametrization variable being ϕ_1. Following [10], we want to compute the conditional probability $\mathcal{P}(p_0,\Delta p_0|\phi_2^0,\Delta\phi_2^0)$ of measuring the value p_0 in the interval Δp_0 when the clock ϕ_2 has some value ϕ_2^0 in the interval $\Delta\phi_2^0$. Thereby, we are removing any reference to the deparametrization variable ϕ_1 and building a gauge-invariant quantity. The conditional probability reads

$$\mathcal{P}(p_0, \Delta p_0 | \phi_2^0, \Delta\phi_2^0) \equiv \lim_{\tau\to\infty} \frac{\int_{-\tau}^{+\tau} d\phi_1 \,\mathrm{Tr}\left[P_{p_0}(\phi_1) P_{\phi_2^0}(\phi_1) \rho P_{\phi_2^0}(\phi_1)\right]}{\int_{-\tau}^{+\tau} d\phi_1 \,\mathrm{Tr}\left[P_{\phi_2^0}(\phi_1)\rho\right]}, \tag{5.9}$$

ρ being the density matrix, and $P_{p_0}(\phi_1)$ the projector on the eigenspace associated with the eigenvalue p_0 at time ϕ_1, similarly for $P_{\phi_2^0}(\phi_1)$. It is more convenient to work in the Schrödinger picture, in which the states evolve in time and the projectors are fixed. In the Schrödinger picture, the conditional probability (5.9) reads

$$\mathcal{P}(p_0, \Delta p_0 | \phi_2^0, \Delta\phi_2^0) \equiv \lim_{\tau\to\infty} \frac{\int_{-\tau}^{+\tau} d\phi_1 \,\mathrm{Tr}\left[P_{p_0} P_{\phi_2^0} \rho(\phi_1) P_{\phi_2^0}\right]}{\int_{-\tau}^{+\tau} d\phi_1 \,\mathrm{Tr}\left[P_{\phi_2^0}\rho(\phi_1)\right]}, \tag{5.10}$$

We introduce the following notations:

$$\begin{aligned} \langle p, \phi_2, \phi_3 | \Psi, \phi_1\rangle &\equiv |p|^{-1/2}\Psi(p, \phi_1, \phi_2, \phi_3), \\ \langle p, \phi_2, \phi_3 | p', \phi_2', \phi_3'\rangle &\equiv \delta(p - p')\delta(\phi_2 - \phi_2')\delta(\phi_3 - \phi_3'), \\ \int dp \int d\phi_2 \int d\phi_3 \, |p, \phi_2, \phi_3\rangle\langle p, \phi_2, \phi_3| &\equiv \mathrm{Id}. \end{aligned}$$

The projectors are given by:

$$\begin{aligned} P_{p_0} &= \int_{p_0-}^{p_0^+} dp \int d\phi_2 \int d\phi_3 \, |p, \phi_2, \phi_3\rangle\langle p, \phi_2, \phi_3|, \\ P_{\phi_2^0} &= \int dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, |p, \phi_2, \phi_3\rangle\langle p, \phi_2, \phi_3|, \end{aligned}$$

where $p_0^\pm = p_0 \pm \Delta p_0/2$, and $\phi_2^{0\pm} = \phi_2^0 \pm \Delta\phi_2^0/2$. The trace in the denominator of Eq. (5.10) is

$$\begin{aligned} &\mathrm{Tr}\left[P_{\phi_2^0}\rho(\phi_1)\right] \\ &= \mathrm{Tr}\left[\int dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, |p, \phi_2, \phi_3\rangle\langle p, \phi_2, \phi_3|\rho(\phi_1)\right] \\ &= \int dp' \int d\phi_2' \int d\phi_3' \, \langle p', \phi_2', \phi_3' \int dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, |p, \phi_2, \phi_3\rangle\langle p, \phi_2, \phi_3|\rho(\phi_1)|p', \phi_2', \phi_3'\rangle \\ &= \int dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, \langle p, \phi_2, \phi_3|\rho(\phi_1)|p, \phi_2, \phi_3\rangle \\ &= \int dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, |p|^{-1}|\Psi(p, \phi_1, \phi_2, \phi_3)|^2, \end{aligned}$$

where we have used the fact that $\rho(\phi_1) = |\Psi\rangle\langle\Psi|$ to obtain the last equality. Similarly, we can show that the trace in the numerator is

$$\mathrm{Tr}\left[P_{p_0} P_{\phi_2^0} \rho(\phi_1) P_{\phi_2^0}\right] = \int_{p_0-}^{p_0^+} dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, |p|^{-1} |\Psi(p, \phi_1, \phi_2, \phi_3)|^2.$$

Finally, the conditional probability (5.10) reads

$$\mathcal{P}(p_0, \Delta p_0 | \phi_2^0, \Delta\phi_2^0) = \frac{\int_{-\infty}^{+\infty} d\phi_1 \int_{p_0-}^{p_0^+} dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, |p|^{-1} |\Psi(p, \phi_1, \phi_2, \phi_3)|^2}{\int_{-\infty}^{+\infty} d\phi_1 \int dp \int_{\phi_2^{0-}}^{\phi_2^{0+}} d\phi_2 \int d\phi_3 \, |p|^{-1} |\Psi(p, \phi_1, \phi_2, \phi_3)|^2}.$$

We ended up with a general expression for the conditional probability giving the value of the triad (or equivalently the volume) when one scalar field is used for deparametrization and an other one as an internal clock. The fact that we use real physical clocks subjected to quantum fluctuations has been incorporated in the model, but we would like to carry on the calculation. This might be a little bit tricky since the evolving observables are not necessarily self-adjoint and therefore the projectors corresponding to different eigenvalues are not generally orthogonal.

VI. CONCLUSION AND PERSPECTIVES

Throughout this work, I have reviewed some issues related to the problem of time in general relativity, quantum gravity and quantum cosmology. In particular it has been shown how the gauge theoretical formulation of general relativity relates to the problem of time and the definition of evolution. We have given the definition of a dynamical system without time and seen how it is possible to recover the notion of time and evolution via Rovelli's evolving constants of motion. At the quantum mechanical level, we have seen that the quantum evolving constants may not be well-defined, thereby restricting the possible choices of time functions. Since we loook for an appropriate framework to study a closed system like the whole Universe in quantum cosmology, there is a need to suppress any reference to parameters which are external to the system. In the case of the evolving constants of motion, we have seen that we can abolish all reference to the external parameter by using the conditional probabilities for Dirac observables. It has been shown that this formalism allows to take into account the quantum fluctuations of the clock and to define the evolution of the system exclusively with relations between physical states. Finally, we have tried to apply these ideas to a simple model of quantum cosmology. A symmetry-reduced model based on the WDW quantization of four dimensional gravity in which matter is sourced by three scalar fields has been introduced.

It appears that the formalism of conditional probabilities with Dirac observables might be an elegant and interesting way to solve the problem of time in quantum gravity. In particular, it relates to many deep philosophical and technical issues such as the loss of unitarity in quantum mechanics and decoherence. Since it provides a way to treate real physical clocks as a part of the system, it would be of great interest to continue this work in the following way: explicitly compute the conditional probabilities; investigate the resulting theory with a semi-classical state; extend these results to LQC and to more general matter fields.

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