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Non-linearities in the quantum multiverse

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Abstract

It has been recently proposed that the multiverse of eternal inflation and the many-worlds interpretation of quantum mechanics could be identified, yielding a unique level of parallelism and offering a new view on the problems of measure and measurement. In the present document, we first argue that the existence of non-linear observables in the quantum multiverse would be an obstacle to this unification in its current form, and, in a second time, justify the presence of these observables based on previous work done in quantum cosmology.

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Introduction

We will start by introducing the not so common and exotic subject of the multiverse, a physical object that possesses its diversity as it appears with different shapes in various branches of physics.

Then, we will give a detailed description of two unifying proposals that aim at identifying two facets of this object.

In a second part, we will review a general proposal of a framework adding non-linear observables and non-linear wave-function dynamics to quantum mechanics and present direct interesting implications.

In the last part, we will focus on the dynamics of quantum cosmological models, where quantum gravitational effects may imply non-linearities and scale-mixing which are both obstacles to the previous patchings.

1 Unifying the multiverses

1.1 The multiverse: universal but diverse

The concept of multiverse has been growing in popularity and has been repeatedly discussed over the past forty years. Although its name implies that it is the most universal physical object, it is also really diverse, having appeared in various branches of physics through different forms:

- It first appeared in the 1950's in Hugh Everett's PhD thesis on the interpretation of quantum measurement [1]. According to this new picture, during the measurement process, the universe splits into as many universes as there are possible outcomes and each one of these is realized in one of the new universes, the set of all these universes or all the branches of the wave-function are the multiverse. This interpretation pairs up quite well with the work done afterward on decoherence: this scheme explains in a satisfactory way the selection of a preferred basis by the environment and the emergence of a classical world to the observer, even though a decohered system still looks like a system in a classical superposition of definite outcomes and it does not explain why the observer only sees one of them (see Annexe 1);

- In the 1970's, cosmology was plagued by several observational paradoxes: extrapolating in the past the size of our universe was of the order of $1\ \mu m$ at $t = t_{planck}$ thus 10^{29} orders of magnitude larger that what we would naturally expect; homogeneity and isotropy of the universe implied that all the matter and radiation content of the Hubble horizon should have been in causal contact in the past, which is not true in a Friedmann-Robertson Walker (FRW) space-time; the flatness of our universe which asks for unnatural initial conditions (extremely low curvature *at the beginning*); the absence of monopoles which should have appeared abundantly during the symmetry breaking of a GUT as the universe was cooling down; and, still not discovered at the time, anisotropies in the CMB at 10^{-5} level. All those puzzles have been solved by adding an era of cosmic inflation: an accelerated expansion sourced by a scalar field, the inflaton. However, this is only consistent in a first de-Sitter, eternally expanding, space where the inflaton starts in a false vacuum and does not decay to the true minimum everywhere at the same time (see Fig. 1). Thus, each bubble arising from a first order phase transition possesses a Hubble horizon [2] and can be seen as a universe on its own. Accepting this scenario of inflation means accepting this picture of a multiverse;

- The most recent reference to a multiverse appears in String Theory, the theory of everything and the most popular candidate to a theory of quantum gravity. In this framework, one difficulty of describing low energy physics, relatively to the Planck scale, is the appearance of an outrageously huge number of vacuum candidates, at least 10^{500}, with, so far, no arguments to discriminate between them and to explain why a particular element of this set has been chosen. The multiverse is a possible answer: none of these vacua is special, they are all realized *in Nature*. One appealing aspect of this reasoning is that it would solve at the same time the problem of the cosmological constant by anthropic arguments: the universes where the physical constants would be much different from our own would be sterile, making the apparition of life and any intelligent being capable of measuring them impossible.

This concept is also a source of controversy, one could advocate that our universe being causally disconnected from the others implies that the multiverse hypothesis cannot be disproved by experiment, making it a non-scientific hypothesis according to the Popper refutability criterion.

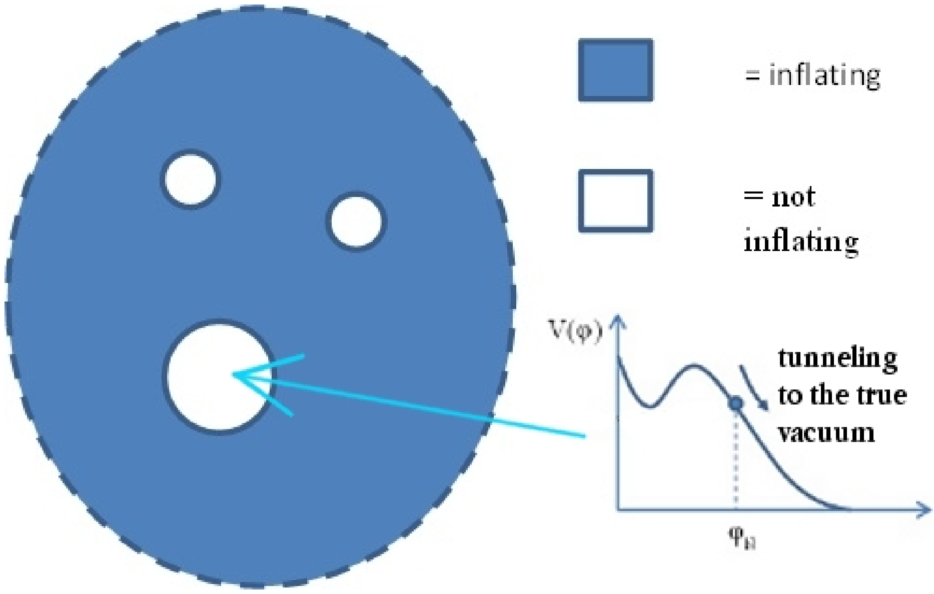

Fig. 1. A schema representing the nucleation of bubble-universes in an inflating *multiverse*

1.2 One multiverse to rule them all

Probably inspired by the fruitful and successful examples of unification paving the history of physics, Bousso and Susskind recently proposed that the multiverse of eternal inflation and the many-worlds from Everett's interpretation of quantum mechanics could be two members of the same elephant, or in other words, two complementary views on the same entity [3].

Their argumentation starts with one concern: in classical cosmology, admitting the scenario of inflation described above, how can one speak about the nucleation of a bubble-universe at specific time and place, when this process is driven by quantum-mechanical laws implying that the field is always in a superposition of false and true vaccua? And one needs this prediction to further describe, for instance, the creation of matter during the reheating, or domain walls during the collision of bubbles, two relics of the inflation era that could, supposedly, be observable 13.7 million years after the big bang. It is a specific case of an issue of semi-classical gravity (ie. describing classical gravity sourced by quantum matter), if our stress-energy tensor is in a superposition of two states corresponding to being localized at two distinct spots, the field equation will force the metric to be in a superposed state, leading to an inconsistency as we chose to describe gravity classically. This flaw would be circumvented in a fully quantum picture, a quantum theory of gravity which still needs to be found. Thus, one needs to describe the nucleation in a classical manner, getting rid of the superpositions by forcing the wave-function to collapse. The current status of quantum mechanics describes this transition from quantum to classical by the mean of decoherence (see Annexe 1). However, to use this scheme one needs to introduce an observer, describing the system by its point of view, and degrees of freedom he cannot monitor, the environment responsible for the emergence of the classical world. The core of Bousso and Susskind's proposal is in the choice of these two elements.

First, one may be uncomfortable with the use of a local description, ie. observer attached, in contrast with the usual impersonal description of large space-time regions encountered in cosmology. However, it is a direct consequence of the postulates of quantum mechanics in a system containing a black hole. Let's look at an object falling into the black hole. From the point of view of an observer outside the black hole at all times, as the object crosses the horizon, the information it contained is stored on the surface of the horizon, according to the results derived in black hole thermodynamics. Now, from an observer falling with the object, nothing special happens when crossing the horizon, as they are in *inertial motion*, and then, they both fall to hit the singularity. Quantum mechanically, if one tries to patch the two points of view, he will see the information xeroxed at the horizon, one copy staying on the surface and the other one entering it. It would be a non-unitary process and violate the non-cloning theorems of quantum mechanics. Thus, to avoid this problem, one needs to renounce to a global description, choosing one point of view, because, by causality, no observer can see both copies of the information. Furthermore, if a global description must be rejected in this specific case, it should be also in general. This conjecture is called Black-Hole Complementarity.

For an observer following an inextendible geodesic starting at a point p and ending at a point q, its causal diamond $\mathcal{C}$ is the volume defined as $\mathcal{C} = I^+(p) \cap I^-(q)$ where $I^-(q)$ is the set of point that can causally influence q and $I^+(p)$ is the ones that can be influenced by p. Or putting it more intuitively, the causal diamond of an observer is the largest set of points that can be causally connected to him (see Fig. 2), and so the minimum space-time volume we need to take into account to have a consistent local picture.

This sets the description used in the proposal, and now, one must chose the environmental degrees of freedom that will be traced over to perform decoherence. The degrees of freedom that escape the future boundary of the causal diamond, labeled as B on the left schema of Fig. 2, will never interact again with our observer, therefore making a natural choice of *environment*: interacting over them as they cross the boundary of the diamond would lead to an exponential suppression of the coherences. This process happens continuously, consider a fiolation of the diamond by three-dimensional hypersurfaces labeled by the proper time of the observer β, it would divide the boundary into two parts B^+ and B^-, the physics seen by our observer holds in the density operator, which is now a function of β: $\rho(\beta)$. Following its geodesic we will integrate out more and more degrees of freedom, and, as a consequence, the sequence of density matrices $\rho(\beta_1)$, $\rho(\beta_2) \ldots \rho(\beta_n)$ will describe a branching process, pictorially a tree.

This object is exactly the wave-function from the many-worlds interpretation of quantum mechanics. The proposal holds on that: if one admits the scenario of eternal inflation driven by quantum tunneling and the interpretation of the collapse of the wave-function by the MWI, then these two layers of parallelism can be identified, as much as bubbles are nucleating in the multiverse, the wave-function is splitting.

However, two weaknesses in the precedent reasoning can be pointed out:

- It does not solve the problem of bubble-universe being superimposed in the past section of the diamond, labeled as $\tilde{B}$ on Fig.2, no degrees of freedom are intrinsically un-monitorable and should naturally be traced over, one would have to find new *environment* components to play this role;

- The separation of the Hilbert space on the future boundary between B^+ and B^- is highly questionable: the boundary of the causal diamond being a null hypersurface, massless operators do not commute at null interval and light-rays, for example, could propagate along it. One could think of integrating *just after the boundary*, but it is not part of the diamond and thus absent from our description.

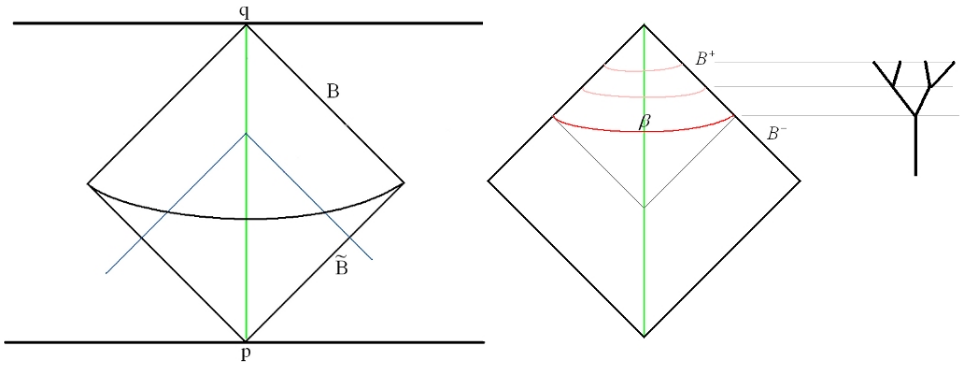

Fig. 2. On the left, the representation of a causal diamond, and on the right, the ongoing process of decoherence, and the branching of the wave-function

Another proposal aims at doing the same unification, it is authored by Nomura [4]. As the previous argumentation, it is built on a local description, on the grounds of the Black-Hole Complementarity but not only.

The multiverse, as a quantum object, is described by a state $|\Psi\,(t=0)\rangle$ starting at $t=0$ in a metastable vacuum, and evolves into different multiverse-states corresponding to different configurations of nucleations. By analogy with the evolution of asymptotic states in high-energy scattering experiments where a transition takes the form:

$$|e^+e^-\rangle \to \alpha|e^+e^-\rangle + \beta|\mu^+\mu^-\rangle + \gamma|e^+e^-e^+e^-\rangle + \dots \tag{1}$$

our previous multiverse-state branches in the same way in a super-position of states corresponding to different nucleating processes, ie:

$$|\Psi(t=0)\rangle \to \sum_i c_i(t)|\,\text{cosmic history i}\,\rangle \tag{2}$$

These multiverse-states are defined on the following Hilbert space:

$$\mathcal{H} = \bigoplus_{\mathcal{M}} \mathcal{H}_{\mathcal{M}} \qquad \text{where} \qquad \mathcal{H}_{\mathcal{M}} = \mathcal{H}_{\mathcal{M},\,\text{bulk}} \otimes \mathcal{H}_{\mathcal{M},\,\text{horizon}}\ , \tag{3}$$

where the sum is taken over 4-dimensional classical geometries, and, is inspired on the Holographic Principle according to which the dynamics of the degrees of freedom in the bulk and on the horizon are relevant for a consistent description. However the geometric separation between the bulk and the boundary is observer dependent, for instance in a DeSitter space, performing a translation will mix the old bulk with the new horizon and vice-versa, see Fig. 3. And, from that, one needs to fix a coordinate chart with its origin to do this splitting; this is equivalent to selecting an observer, thus, the local description is not only a consequence of the BHC but also a way to get rid of the redundancies of general relativity.

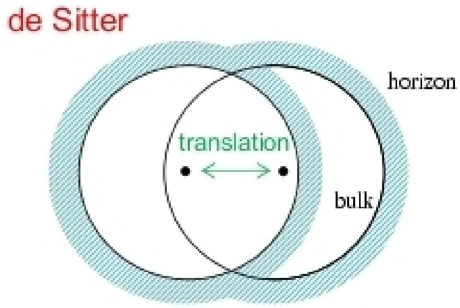

Fig. 3. Observer dependance of the horizon

Having defined the physical space of the theory, one can now ask about the way to compute predictions. In its current form, eternal inflation is a framework lacking predictivity: in a future-eternally inflating space-time an event that can happen once will happen an infinite amount of times. This failure is called the measure problem of eternal inflation. Suppose one is interested in computing the occurrences of two events A and B having non-vanishing probabilities to appear, he will find N_A, $N_B = \infty$, but he may hope that the conditional probability of an event occurring knowing another one did might be finite, probably under the limit of a certain regulator. This takes the explicit form of a *generalized* Born rule giving the probability of an event B to happen knowing that A happened (or will happen, the author including postdictions, hence the name *generalized*):

$$P(B|A) = \frac{\int_0^{+\infty} dt \, \langle\psi(0)|U(0,t)\mathcal{O}_{A\cap B}U(t,0)|\psi(0)\rangle}{\int_0^{+\infty} dt \, \langle\psi(0)|U(0,t)\mathcal{O}_{A}U(t,0)|\psi(0)\rangle} \tag{4}$$

where $U(0,t)$ is a unitary evolution operator, $\mathcal{O}_A$ and $\mathcal{O}_{A\cap B}$ are projection operators and t being a natural regulator. Some computations have actually been performed, for instance, the number of universes with a CMB temperature of 3K seen by an observer who has, in its past light-cone, a bubble having nucleated and now a CMB at 2.725K is equal to $10^{10^{59}}$. This immenseness can be explained: as an inflating space expands the more nucleation it encompasses *on a time slice* leading to imbricated exponentials. One could argue that these huge numbers are difficult to interpret, replacing the measure problem by the youngness paradox.

It is assumed that every prediction in quantum mechanics should take the form of (4) where the projectors $\mathcal{O}_{\mathrm{X}}$ now select a specific measurement configuration $A(t)$ and an expected output B, both observables. Therefore (41) puts on the same level the predictions of seeing a specific cosmology nucleating in a inflating universe or a low-energy quantum experiment in an universe which has cooled down. The history of the wave-function is represented by Fig. 4., it goes through phases of splitting and amplification (decoherence). On the grounds of our generalized formula, cosmological and quantum mechanical questions receive the same treatment, therefore it leads us the conclusion that the two multiverses are the same.

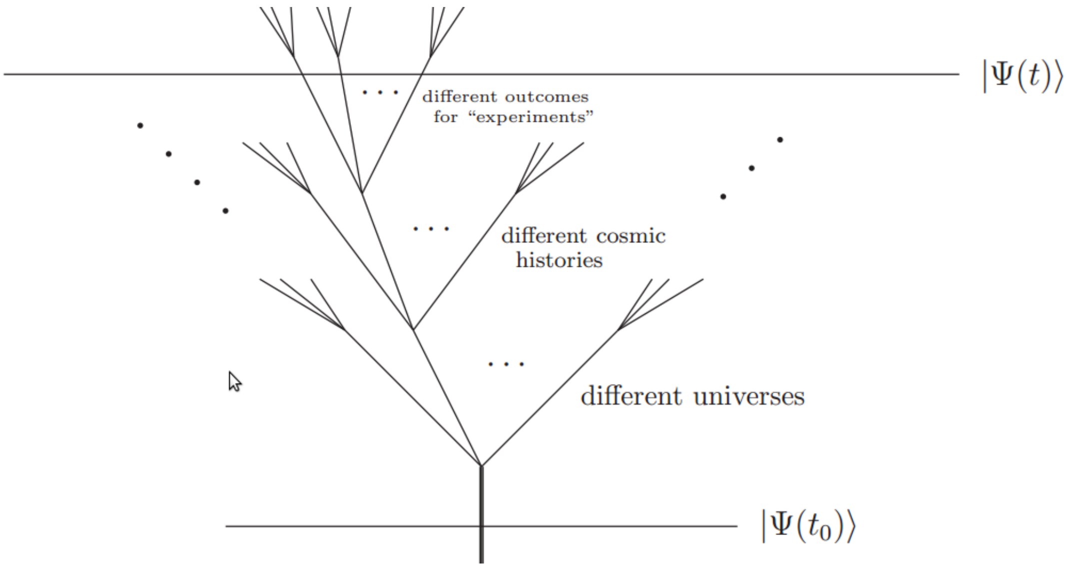

Fig. 4. The wave-function enduring phases of amplification and branching, on the largest and the smallest scales.

One point should be stressed about the decomposition of the Hilbert space (3), as previously said it is motivated by analogy with the Fock spaces used in quantum field theory, the space containing the eigenvectors of the free Hamiltonian:

$$\mathcal{H} = \bigoplus_{\mathrm{n}=0}^{\infty} \mathcal{H}_{1\mathrm{P}}^{\otimes \mathrm{n}} \tag{5}$$

where $\mathcal{H}_{1\mathrm{P}}$ is the Hilbert space of a one-particle state, and each element of the sum represents the space where n-particle states live.

However, when the interactions are turned on, one does not know the eigenvectors of the interacting Hamiltonian, and is forced to use the asymptotic theory, thereby renouncing to describe intermediate states. Consequently, the previous decomposition, describing $|\Psi(t)\rangle$ at all times $t > 0$, supposes implicitly that there are no interactions between the elements of the subspaces, ie. between different classical solutions. The presence of interactions would flaw (3). This justifies our following interest in non-linear quantum mechanics and, later, in non-linearities in quantum cosmology.

2 Non-linear quantum mechanics

2.1 A generalized framework

One tacit assumption of the two argumentations described above is the linearity of the quantum framework at all scales, even cosmological ones. Thus, a possible line for discussing the accuracy and the pertinence of their point could be to investigate possible modifications of quantum mechanics in a multiverse, specifically the appearance of non-linear observables.

The idea of generalizing quantum mechanics to describe a wave function evolving non-linearly is probably as old as quantum mechanics itself. Many attempts have been made to solve the measurement problem and explain the collapse of the wave-function by it own dynamics, modifying Schrödinger's equation by adding to the Hamiltonian a dependance on the wave-function itself; but, so far, those approaches have been sterile and decoherence brought a more satisfying answer to those questions.

From now on, we will focus on a peculiar extension of quantum mechanics to a framework holding non-linearities. It has been proposed by Weinberg [5]. In this new formalism, the system is still described by a wave-function ψ_i ($i \in 1, \dots, N$). Usually assuming that observables are linear in quantum mechanics takes the form of the following postulate: *to every observable on a quantum system there exists a linear Hermitean operator acting on the Hilbert space*; which is now relaxed. Any observable is a real function of the wave-function and its conjugate $a(\psi_j, \psi_j^*) \in \mathbb{R}$, it is still not the broadest generalization as we demand to these functions to be of order 1 in ψ and ψ^*; ψ and $\alpha\psi$ ($\alpha \in \mathbb{C}$) should be physically indistinguishable:

$$a = \psi_k \frac{\partial a}{\partial \psi_k} = \psi_k^* \frac{\partial a}{\partial \psi_k^*} \tag{6}$$

In other words, we ask for stability of the solutions under multiplication but not addition, and from there, we lose the superposition principle which was at the core of the linearity in quantum mechanics.

The set of observables forms an algebra where the intern product is defined by:

$$a * b = \frac{\partial a}{\partial \psi_k} \frac{\partial b}{\partial \psi_k^*} \tag{7}$$

Quite evidently, the identity element under this law is the norm of the wave-function $n = \psi_i \, \psi_i^*$. And contrary to the matrix product, this one is not associative and it is one of the main reasons behind the departures from quantum mechanics. Although, similarly with the linear framework, one can choose for the time-dependance to hold on the operators or on the states, the so-called Heisenberg and Schrödinger pictures:

$$\frac{da}{dt} = -i[a, h] = -i(a * h - h * a) \quad \text{for the Heisenberg equation} \tag{8}$$

$$i\frac{d}{dt}\psi_k = \frac{\partial h}{\partial \psi_k^*} \quad \text{for the Schrödinger equation} \tag{9}$$

where, evidently, h is the Hamiltonian.

One can immediately note two conserved quantities: h and n. However, because of non-associativity, $(h*h)*h \neq h*(h*h)$ and powers of h may be not anymore conserved, one cannot guarantee the integrability of the system for $N > 2$ (N, the number of components of the wave-function). In ordinary quantum mechanics, the wave-function would evolve quasi-periodically on a N-dimensional torus, defined on a space spanned by $\mathrm{Re}(\psi_i)$ and $\mathrm{Im}(\psi_j)$; whereas, within this new framework one should expect chaotic dynamics through the same space if $N > 2$. This deviation deserves to be stressed. Still, the most striking consequences of this new formalism come from the study of composite systems when their components are far-separated in space.

2.2 Compositeness and strangeness

One important point to clarify in details is the way our formalism handles the description of composite systems, ie. systems distinguishable in subparts. The goal is to merge two systems I and II, described respectively by ψ_k and ϕ_l and which time-evolutions are generated by $h_I(\psi, \psi^*)$ and $h_{II}(\phi, \phi^*)$, into a unique (meta)system $I + II$ described by a single wave-function Ψ_{kl} and an overall Hamiltonian $H_{I+II}(\Psi, \Psi^*)$.

As a sanity-check, one can verify that in the peculiar case where the two systems are uncorrelated and not interacting with each other:

$$\Psi_{kl} = \psi_k \, \phi_l \tag{10}$$

$$H_{I+II}(\Psi, \Psi^*) = \sum_l h_I(\Psi_{jl}, \Psi^*_{kl}) + \sum_k h_{II}(\Psi_{kl}, \Psi^*_{km}) \tag{11}$$

The sums may seem obscure at first but they would give the right tracing operations if we had added to our hypothesis the bilinearity of the Hamiltonian:

$$h_I(\phi, \phi^*) = \phi^*_l \, h_{lk} \, \phi_k \;\; \text{and} \;\; h_I(\Psi, \Psi^*) = \psi^*_j \, \psi_j \; \phi^*_l \, h_{lk} \, \phi_k \tag{12}$$

is written in linear quantum mechanics as:

$$\langle \Psi | H_I | \Psi \rangle = \langle \psi_I \psi_{II} | H_I \otimes I_{II} | \psi_I \psi_{II} \rangle = \langle \psi_{II} | \psi_{II} \rangle \, \langle \psi_I | H_I | \psi_I \rangle \tag{13}$$

Therefore, from the evolution of the ensemble wave-function:

$$i \frac{d}{dt} \Psi_{kl} = \frac{\partial h_{I+II}}{\partial \Psi^*_{kl}} \tag{14}$$

and using the separability (10) and the homogeneity of the Hamiltonians (11) one derives the expected evolution for both subparts:

$$i \frac{d}{dt} \psi_k = \frac{\partial h_I(\psi, \psi^*)}{\partial \psi^*_k} \;\; \text{and} \;\; i \frac{d}{dt} \phi_l = \frac{\partial h_I(\phi, \phi^*)}{\partial \phi^*_l} \tag{15}$$

Having stressed that, one may now look for intriguing new features appearing in the study of compound systems, for example one could suspect that adding non-linear variables means adding information which could be a source of real EPR violations, beyond the fictitious EPR violations of ordinary quantum mechanics where no physical information is propagated faster than light. This work has mainly been done by Polchinski in [6]. We will describe and emphasize his results in the following.

The first model considers two widely separated systems I and II, that may or may not have been correlated, and described globally by the wave-function Ψ_{iJ}. Making the assumption that the Hamiltonian is only a function of an observable defined on the subsystem I:

$$h_{I+II} = e(t)\; a_I(\Psi_{iJ}, \Psi^*_{iJ}) \tag{16}$$

One can try probing violations of special relativity. Assuming that $e(t)$ is turned off for $t < 0$ and weakly turned on at $t > 0$, the evolution of an observable restricted on the subsystem II will be, at first order in e:

$$a_{II}(t) = a_{II}(0) - i\{a_{II}(0), a_I(0)\} \int_0^t e(t')\, dt' \tag{17}$$

Thus, the perturbation on the first subpart is propagated instantaneously to the second, it could be probed by the mean of the observable a_{II} and used to carry information. This strongly violates special relativity and is the mark of a real EPR violation.

In the special case where a_{II} is a bilinear observable, ie. $a_{II} = \Psi^*_{iJ} A_{JK} \Psi_{iK}$, $\{a_{II}, .\}$ can be seen as the generator of a transformation on the second index of the wave-function. Thus, to rule out the previous violations, one needs:

$$\{a_{II}(0), a_I(0)\} = 0 \tag{18}$$

implying that a_I is a scalar under this transformation, it needs to have no *freedom* under the second index. More explicitly, it should be restricted to be a function of the density matrix partially traced over the second subsystem:

$$a_I \equiv a_I(\sum_J \Psi_{iJ} \Psi^*_{kJ}) \tag{19}$$

We see that the Hamiltonians defined by (11) are not of this form, except when linear, allowing EPR communications in the general case. At this stage, we can state that one needs to be careful and more restrictive in the definition of the observables to not enter in conflict with locality.

One could now ask if observables of the form (19) allow unusual communication. Spoiling the suspense, the answer is yes, and can be highlighted by the following model:

- We consider the union of two quantum systems: a spin-$\frac{1}{2}$ ion and an experimenter handling a quantum coin, simply a quantum object giving two different outputs when solicited ;
- The ion starts by entering a first Stern-Gerlach apparatus along the z-axis, the observer takes note of the value of the spin, and the two paths are rejoined;
- If the value measured was *up*, the observer does nothing. If it was *down*, he flips its *quantum coin*, if the output is heads, he still does nothing (action a); if it is tails he rotates the spin along the y-axis (action b) by coupling the ion with the linear observable $\mathrm{Tr}(\rho^I \sigma^2)$, where ρ^I is the partial density matrix of the ion;

- The ion now enters a region where it is coupled to non-linear observable during the time τ :

$$h_3 = f\frac{\mathrm{Tr}(\sigma^1\rho^I)^2}{\mathrm{Tr}\rho^I} \tag{20}$$

One should note that this observable is of the form (19) and excludes EPR communication;

- At last, if the experimenter saw an *up* outcome in the Stern-Gerlach, he performs a new measurement on the z-component of the ion spin (see Fig. 5), supposedly with a new SG apparatus.

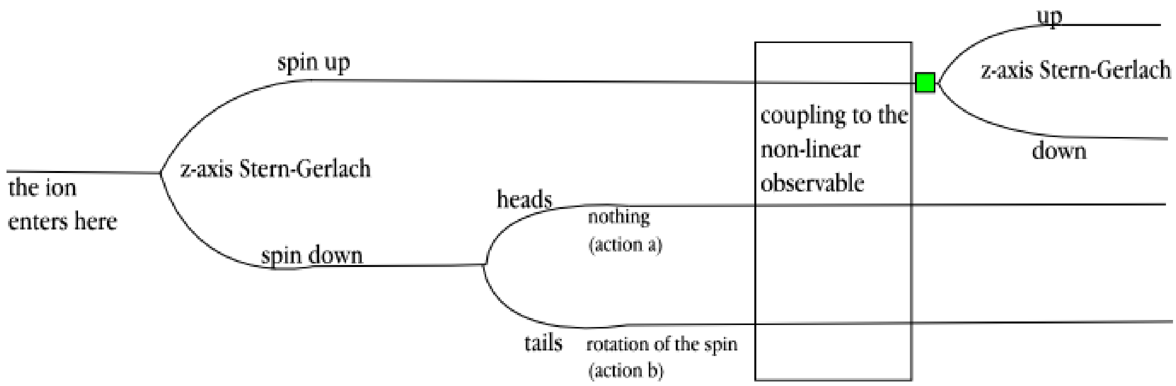

Fig. 5. Polchinski's second model: the density matrices (21) and (22) are calculated at the green point, and depend on the lower branch of the wave-function.

Right before the last step, a calculation of the density matrix of the wave-function that was measured with a spin *up* in the first Stern-Gerlach gives (choosing f in (20) so that $\tau f = \frac{\pi}{2}$):

$$\rho^{I\ up} = \frac{1+\sigma^3}{4} \quad \text{if action } a \text{ was taken} \tag{21}$$

$$\rho^{I\ up} = \frac{1-\sigma^3}{4} \quad \text{if action } b \text{ was taken} \tag{22}$$

Therefore, a new disturbing fact appears, when the experimenter will measure the spin along the z-axis for the second time (after having measured *up* at first), the result will depend on the outcome of the coin: *up* for (21) and *down* for (22); but the coin is only flipped if the first measure gave *down*. Our apparatus is capable of predicting what the experimenter would have done in a non-physically realized situation!

In the Many-Worlds Interpretation, this is an evident case of communication between different branches of the wave-function, also dubbed an Everett-phone. Recalling the propositions of unifying it to the eternally inflating multiverse, it would lead to the inconsistent picture where physics on our universe depends on what has happened in the other universes, turning it impossible to make unambiguous predictions about phenomenas that are continuously influenced by events on every other bubble-universes of the multiverse. However small they could be, the presence of one of these non-linear observables would be a serious obstacle to the patchings proposed in the first chapter, at least in their current form.

These insightful models have outlined exotic behaviors in non-linear quantum mechanics, but, so far, they are totally artificial and one should ask about the experimental evidences of these additions. Experiments probing the amplitude of non-linearities in atomic systems have been performed. It has been looked for detuning phenomenon in radio-frequency transitions of $^9Be^+$, see ref. [7], typically characteristic of a non-linear oscillator. However, this phenomenon has not been observed and the upper bound on the relative magnitude of the non-linear contributions to the linear one is 10^{-21}.

As previously stated, the presence of one non-linear observable, however extremely suppressed it may be at low energies, is enough to flaw the previous proposals of unification. Even if experiments put really stringent bounds on them at energies around the GeV, one could still try to derive them or justify there appearance at higher energies, for example around the Planck scale.

Quantum cosmology is one framework that aims at describing physics around these energies. This is why, in the following, we will argue on quantum cosmological grounds about the presence of non-linearities in the dynamics of the wave-function of the universe, and the wave-function of a restricted subspace of it.

3 Non-linear observables in the quantum multiverse

3.1 Quantum mechanics from quantum cosmology

Built on the canonical Hamiltonian formalism to quantize gravitation, quantum cosmology describes the universe by a wave-function $\Psi(h_{ij}(\vec{x}), \phi(\vec{x}))$ (see Annexe 2 for more details). More precisely Ψ is a wave-functional whose function-variables belong to the so-called superspace (totally unrelated to supersymmetry): the space of all 3-metrics and matter configurations on an hypersurface. The dynamics of this wave-function are constrained by the Wheeler-DeWitt equation, equivalent to the 00-component of Einstein's field equation:

$$H\Psi = 0 = i\frac{d}{dt}\Psi \tag{23}$$

Thus, one could more precisely speak about absence of dynamics. Quite intuitively, it is a direct consequence of the absence of clock exterior to the universe. To this day, no solution of this equation has been found, and to extract information from it, the sharpest tool is the mini-superspace approximation: all the variables of the superspace are frozen to the exception of a finite number, passing from a problem of quantum field theory to a problem of quantum mechanics. We will restrain ourselves to this approximation from now on.

Our ambition is to derive non-linear corrections to quantum mechanics from quantum cosmology, but to this goal, one needs to prove that (23) holds the dynamics of quantum systems, ie. Schrödinger's equation, when looking at small quantum systems. This result has been derived in [8] under specific assumptions. Starting by looking at a model consisting of a set of variables h^α forming the n-dimensional mini-superspace whose metric is $g_{\alpha\beta}$. It is described by the following action:

$$S = \int dt\, [p_\alpha \dot{h}^\alpha - N(t)(g^{\alpha\beta} p_\alpha p_\beta + U(h^\alpha)] \tag{24}$$

where p_α is the conjugate momentum of h^α and $U(h^\alpha) = \sqrt{h}(V(\phi) - {}^3R)$ the superpotential. The WdW equation is derived as a constraint equation imposed by the Lagrange multiplier $N(t)$:

$$H_0\Psi = (\lambda\nabla^2 - \frac{1}{\lambda}U)\Psi = 0 \tag{25}$$

with $\nabla^2\Psi = \nabla_\alpha\nabla^\alpha\Psi = \frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}\, g^{\alpha\beta}\partial_\beta)\Psi$ and $\lambda = 16\pi M_p^{-2}$ with M_p the Planck mass.

Dirac's program of quantizing constrained systems consists in, first, solving the constraints equations at the classical level and then imposing the commutation relations on the dynamical variables). Thus, at this stage, one should quantize the variables h^α. But this model is more subtle, only a subpart of the variables will be assumed to be quantum, and the other semi-classical. We will denote the first ones by q^μ and the second ones by c^α.

Furthermore, we will look at a WKB expansion not in $\hbar$ but in the parameter λ, representing the fact that the universe should look classical when its size is much larger than the Planck length (or volume). In some way, it is the requirement that a macroscopic universe should be describable by classical equations. For instance, the dynamics of the inflaton should be dominated by its quantum fluctuations in the first moments of the universe and then by a classical evolution once the universe has achieved a macroscopic size.

The Wheeler-DeWitt equation becomes:

$$H\,\Psi(q^\mu, c^\alpha) = (\lambda\nabla_0^2 - \frac{1}{\lambda}U_0 - H_q)\Psi(q^\mu, c^\alpha) \tag{26}$$

where ∇_0^2 is the part of ∇^2 of order 0 in λ (the higher order terms would be neglected in the next steps of the calculus), U_0 is independent of q^μ and H_q is of order -1 in λ.

We inject the following form for Ψ ($A,\, S \in \mathbb{R};\, \Phi \in \mathbb{C}$) :

$$\Psi(q^\mu, c^\alpha) = \Psi_0(c^\alpha)\Phi(q^\mu, c^\alpha) = A(c^\alpha)e^{iS(c^\alpha)}\Phi(q^\mu, c^\alpha) \text{ with } (\lambda\nabla^2 - \frac{1}{\lambda}U)\Psi_0 = 0 \tag{27}$$

We assumed the separability of the wave-function between the subsystem and *the background*, and in addition the independence of Ψ_0 on q^μ; in other words, the effects of the small quantum system on the universe are neglected. To first order in λ one gets:

$$2i(\nabla_0 S)(\nabla_0 \Phi) = H_q \Phi \tag{28}$$

and using Hamilton's equation:

$$\dot{c}^\alpha = 2N\,\nabla^\alpha S \tag{29}$$

ends with:

$$i\frac{\partial\phi}{\partial t} = N H_q \phi \tag{30}$$

In the gauge $N = 1$, we recover the familiar Schrödinger equation for the quantum subsytem.

Quantum cosmology holds quantum mechanics on the small scales. A time and more precisely a time-evolution is recovered with the use of the semi-classical variables c^α. Indeed, one needs a classical background to use a classical clock, in the sense that it would not be subject to quantum fluctuations.

In [9], the development has been performed to the order λ^2, giving, in the same gauge:

$$i\frac{\partial\phi}{\partial t} = H_q\phi - \lambda^2\,\frac{\nabla^2(A\phi)}{A} \approx H_q\phi - \lambda^2\,(\frac{\nabla^2 A}{A}\,\phi + \nabla^2\phi) \tag{31}$$

where the last step is taken on the assumption that the variation of A ($\nabla^\alpha A \approx 0$) is weak on times scales relevant to the study of the subsystem. At first sight, this equation is as much linear in ϕ as (30), but it presents a weak source of non-linearity: the equation of evolution of $A(c^\alpha)$ may depend on the quantum variables through the potential $V(\phi)$. Furthermore, the off-diagonal terms in the metric of the superspace between classical and quantum variables where supposed to be of order λ and, at order λ^2, one should expect more retro-action of $A(c^\alpha)$ on Φ from the Laplacian term. However, it is a feeble non-linear perturbation to the dynamics of ϕ and does not take the shape of the linearities accounted by Weinberg.

3.2 A third quantization scheme

One flaw of quantum cosmology is the fact that probabilities are ill-defined in the most general case. For instance, one could look at (25) as a Klein-Gordon equation with a mass depending on the position in space-time, and from that, deduce the conserved current:

$$j^\alpha = -\frac{i}{2}g^{\alpha\beta}(\Psi^*\nabla_\beta\Psi - \Psi\nabla_\beta\Psi^*) \tag{32}$$

However, as in the Klein-Gordon case, its probability density, the 0-component, is not positively defined which is nowadays interpreted as the prediction of anti-matter. Even worst, the signature of the metric on the superspace may change from one model to another impeding a general probabilistic interpretation. For each case, one is forced to chose specific hypersurfaces on the mini-superspace giving positive probabilities, $dP = j^\alpha d\Sigma_\alpha$ where $d\Sigma_\alpha$ is the infinitesimal surface element, and a current conservation equation looking like:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{j} = 0 \tag{33}$$

where the role of t needs to be played by one of the mini-superspace variables.

This has led to the supposition that Ψ should not be looked at as a wave-function but as an operator creating and annihilating universes, reusing the old recipe of quantum field theory. This scheme is called third-quantization, a step similar to the transition from quantum mechanics to quantum field theory in the description of a system of particles.

One of this model [10] has been proposed to solve the cosmological constant problem and has attracted our attention for several reasons. The author starts with the following effective action:

$$\Gamma\{\Phi\} = \Phi(g_1)(\mathcal{H}(g_1, g_2) + \Phi(g) \star V[g|g_1, g_2])\Phi(g_2) \tag{34}$$

where $\Phi(g)$ is the third quantized field whose role has been described above, g is the variable carrying the metric and matter degrees of freedom (every repeated index is functionally integrated over), $\mathcal{H}$ is the Wheeler-DeWitt Hamiltonian. It contains a trilinear term V motivated by processes of bifurcating universes, one universe splitting into two, similarly to a ϕ^3 vertex in quantum field theory. On top of that, the assumption that those process only appear on small scales is made. Or, in equation, if we split Φ in two, $\Phi = \Phi_S + \Phi_L$ where the first has support on small scales and the second on large ones, we have:

$$\Phi \star V = \Phi_S \star V \tag{35}$$

One other feature of this model is the mixing of scales, still through the trilinear term. Ψ_L and Φ_S are highly co-dependent, as shown by the classical field equations:

$$(\mathcal{H} + \Phi_S \star V)\,\Phi_L = 0 \tag{36}$$

$$(\mathcal{H} + \Phi_S \star V)\,\Phi_S = -\frac{1}{2}\Phi[g_1]V[g|g_1, g_2]\Phi[g_2] \tag{37}$$

(36) is easily interpreted with renormalization arguments, the physics on the I.R. depends of the physics on the U.V. through the redefinition of the coupling constants of the Lagrangian or Hamiltonian, as if we had integrated out the higher energies degrees of freedom. (37) contains a source term playing the role of a cosmological constant $-\frac{1}{2}\Phi_L[g_1]V[g|g_1, g_2]\Phi_L[g_2]$ which will further offer a mechanism generating an exponentially suppressed value for the cosmological constant, thus a possible answer to this puzzle.

Furthermore, this influence of the physics on large scales to the physics on short distances is an obstacle to the separation of the degrees of freedom between the bulk and the horizon infinitely far away.

In addition to this, the very existence of the bifurcating processes implies interactions between several geometries, thereby ruining the separation (3) *à la Fock*. However, the trilinear term added would not give an observable of the kind proposed by Weinberg, ie. satisfying

(6); for the reason that the previous condition imposes the conservation of the norm of the wave function and, in a model where universes would be created and annihilated this is clearly violated. Indeed, those processes have been proposed to explain the loss of coherence under gravitational dynamics as it was defended by Hawking; until recent years, and the proof that gravity does not *burn information*. Nonetheless, the model proposed in [10] is still pertinent as (34) is a general action derived from string field theory as argued in [11].

Conclusion

Although the idea of unifying the MWI and the eternally inflating multiverse is seducing, as it would bring a new unification step into physics and propose a natural way of regulating divergent cosmological probabilities; it is only consistent under the strong hypothesis that the parallel branches do not communicate and interact with each other. Nevertheless, two obstacles to this identification can be raised from quite general arguments: at high energies coupling between the universes may appear and the mixing of scales could be a major ingredient to solve the cosmological constant problem. Even if we did not manage to derive or justify the presence of strong non-linearities as the ones accounted in [5] and exhibiting the disturbing behaviors exhibited in 2.2, our qualitative arguments raise serious doubts on the idea that it suffices to describe the dynamics of a single observer to apprehend the physics of the multiverse.

Annexes

1 Decoherence or how the quantum world can appear classical to an observer

To describe the emergence of a world governed by classical physics (ie. no superpositions or interferences) to our observer, we have made repeatedly use of the process of decoherence. We will now present a description of the subject and its implementation in the framework of quantum mechanics.

Since its beginning, quantum mechanics has clashed with our direct experience of the world: no sane person will ever tell you that they have already been in two distinct places at the same time or that they had the experience of interfering with a colleague. One of the conceptual difficulties has been to describe the process of measurement on a quantum system. Bohr's first proposal was to suppose the existence of macroscopic classical apparatus and use them to probe quantum systems, thereby implying that the quantum theory was not universal and setting an arbitrary cut between the *quantum* and the *classical* world, forcing the physicist to use two frameworks and conciliate them in some extent. On top of being conceptually unsatisfying this boundary is ruled out by many examples: for instance, a macroscopic gravity-wave detector must be treated as a quantum oscillator even though it weights a ton. Moreover, one could think of an experimental setup where the condition of a macroscopic object depends on the state of a microscopic system, as in the provoking example of Schrodinger's cat: the life of the animal depending on the decay of an atom, a typically quantum process, therefore the poor animal being in a superposition of dead and alive conditions.

Later, Von Neumann started treating the apparatus $\mathcal{A}$ as a quantum object, which would become entangled with the observed system $\mathcal{S}$, in the following way for a two-dimensional case:

$$(\alpha|\uparrow\rangle_{\mathcal{S}}+\beta|\downarrow\rangle_{\mathcal{S}})\,|\circ\rangle_{\mathcal{A}} \longrightarrow \alpha|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{A}}+\beta|\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{A}} \qquad \text{with } \ |\alpha|^2+|\beta|^2=1 \tag{38}$$

This (unitary) step is called pre-measurement, basically demanding that the apparatus carries the information about the state of the system, which is the least we can ask for. Described by its density matrix the global systems looks like:

$$\rho^c_{\mathcal{SA}} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \tag{39}$$

Everyone knows that the outputs of the measurement would be $\uparrow$ and $\downarrow$ with probabilities $|\alpha|^2$ and $|\beta|^2$ respectively, and thus, the information holding the outcomes could be handled by the following density matrix:

$$\rho^r_{\mathcal{SA}} = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \tag{40}$$

We got rid of the coherences, as are called the non-diagonal terms of the density operator. However the transition from ρ^c to ρ^r is non-unitary. This evolution from a pure state to a mixed one cannot result from a quantum evolution described by the Scrödinger equation, the purity ($\mathrm{Tr}\,\rho^2=1$) is conserved in time if the time evolution operator, $U(t_1,t_2)=\mathrm{e}^{iH(t_1-t_2)}$, is unitary; putting it in other words: unitarity condemns the system to purity. The information about the correlations has been lost. One can quantify this lost by the Von Neumann entropy $\mathcal{H}=-\mathrm{Tr}(\rho\ln\rho)$, which increased from 0 (pure state) to $-(|\alpha|^2\ln|\alpha|^2+|\beta|^2\ln|\beta|^2)$. The wave function collapse *à la Von Neumann* is precisely this transition, a non-unitary step that must be added ad hoc to the postulates of quantum mechanics. Furthermore, one more flaw of this procedure is the arbitrariness in the definition of the possible outcomes: getting back to our cat example, why are the two observed states $\{|\text{alive}\rangle\ ;|\text{dead}\rangle\}$ and not $\{\frac{1}{\sqrt{2}}(|\text{alive}\rangle+|\text{dead}\rangle)\ ;\frac{1}{\sqrt{2}}(|\text{alive}\rangle-|\text{dead}\rangle)\}$? They are both two valid bases of our two-dimensional Hilbert space, with nothing so far to favor the first one over the second.

Decoherence is a scheme pushing the previous description further by adding the effect of the environment on the couple apparatus plus system. It appeared in the 1970's with work on open quantum systems, and has seen its major developments done in the early 1980's [12, 13]. Decoherence does not demand to add anything to the postulates of quantum mechanics, nonetheless it is a minor paradigm shift. The environment, which had been, so far, usually seen as a noise or a undesired perturbation on idealistic objects studied by physics, is now at the core of the emergence of classical behavior. Any practical quantum system is open and interacting with a huge number of degrees of freedom that cannot be monitored by the experimenter, and those are precisely the degrees of freedom we are labeling as environment.

Let's suppose that our previous system is now entangled with this *bath* $\mathcal{E}$:

$$(\,\alpha|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{A}}+\beta|\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{A}}\,)\,|E\rangle_{\mathcal{E}}\longrightarrow\alpha|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{A}}|E\uparrow\rangle_{\mathcal{E}}+\beta|\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{A}}|E\downarrow\rangle_{\mathcal{E}} \tag{41}$$

As stated before, the observer cannot monitor the environmental components, thus, his description of the system relies on the partial density operator after tracing over these intrinsically unobservable degrees of freedom. On top of that, assuming that the system is entangled with more and more elements from the bath, $|E\rangle = |e_1\rangle|e_2\rangle\cdots|e_n\rangle$, the quantum information on the observed system is amplified; and in the limit where $n\to\infty$ the overlap of the differently amplified bath states tends to 0, ie. $\langle E\downarrow|E\uparrow\rangle\to 0$. The partial density matrix thus takes the following form:

$$\rho_{\mathcal{S}\mathcal{A}} = \mathrm{Tr}_{\mathcal{E}}\,\rho_{\mathcal{S}\mathcal{A}\mathcal{E}} = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \tag{42}$$

This inherently quantum system in a superposition of states is now seen, by our observer, as a statistical mixture. One should point that:

- Unitarity is conserved on the all quantum system ($\mathcal{S}$ystem + $\mathcal{A}$pparatus + $\mathcal{E}$nvironment), the density matrix $\rho_{\mathcal{S}\mathcal{A}\mathcal{E}}$ stays in a pure state, the information about the quantum correlations is not fundamentally lost, it has *leaked* in the environment, once again the Von Neumann entropy has increased taking account of the fact our observer has lost information by integrating out degrees of freedom;

- The decomposition (41) is unique as stated by the tridecompositional uniqueness theorem. The freedom of rotating the basis, and by the same deciding of the out-coming observables, disappeared. More precisely those observables are selected by the interacting Hamiltonian between the environment and the observed system. Only its eigenstates, dubbed pointer-states, will not be suppressed by the interaction; this feature is called environment induced selection. It implies a more stringent definition of what observables are accessible to the observer, they must commute with the previous Hamiltonian, whereas, forgetting decoherence, we could just pick any set of commuting auto-Hermitian operators.

In Ref. [13], one can find an explicit calculation done for a spin-$\frac{1}{2}$ fermion in a bath of identical particles, where the coherences decay exponentially. Every decohering calculations exhibit this behavior with characteristic times really small, for instance a molecule in the air would have a typical time of decoherence of 10^{-32} s.

This scheme has encountered its first experimental confirmation in 1996 when Serge Haroche and his co-workers observed the progressive loss of coherence of a system of Rydberg atoms induced by the interaction with a coherent low-energy electromagnetic field [14].

The classical world we experience in everyday life emerges from the purely quantum property of entanglement and from our ignorance, or limited knowledge, of the huge number of degrees of freedom that surrounds us. However, one cannot argue that decoherence has solved the measurement problem of quantum mechanics, one missing point, is the selection of a specific outcome from the mixed decohered density matrix. Still, it collaborates pretty well with the MWI of Everett, each one of those outcomes will be realized in a new branch of wave-function, the decoherence process selecting the base of those.

2 A crash course in canonical quantum cosmology

Before giving a quick, far from exhaustive, description of the framework called canonical quantum cosmology, one must first say some words about the approach of quantum gravity behind it. The first serious attempt to conciliate Eintein's general relativity and the quantum theory has been initiated by Paul Dirac in the 40's, but it is only in the beginning of the 60's that this emergent field knew its first breakthrough with the contributions of Bryce DeWitt and John Wheeler. Their canonical approach uses the ADM formalism: on the assumption that our space-time has a non-pathological causal structure, ie. no closed time-like geodesics and containing at least one space-like Cauchy surface allowing an initial value formulation, it is arbitrarily foliated in a family of space-like hyper-surfaces labeled by and artificial time Σ_t. This foliation allows to work with an Hamiltonian formulation of the theory and from there, quantize the dynamical variables and describe their evolution. One could also be tempted to stay in a Lagrangian description, keeping explicitly the invariance under the group of diffeomorphism, but he will run into two technical obstacles:

- as a gauge theory, general relativity is plagued by huge redundancies in its formulation, ie. the former liberty of coordinate redefinition, thus a naive path integral over 4-metrics would sum over physically identical configurations, and the Faddeev-Poppov method of "quotienting" by the symmetry group is ill-defined when it comes to the infinite dimensional group of diffeomorphisms;
- moreover, one may want to add to this path integral a summation over different topologies, taking into account eventual quantum tunneling processes between topologically distinct universes, but the classification of topologically different manifolds in 4 dimensions is still not achieved by mathematicians.

So, one starts with the familiar Einstein-Hilbert action coupled to matter:

$$S = \int_M d^4x \sqrt{-g}\, ({}^4R - 2\Lambda) + S_{\mathrm{matter}} \tag{43}$$

where ${}^4R$ is the Ricci scalar defined on the 4-dimensional manifold M and Λ the cosmological constant. We now perform a 3+1 decomposition of M in $\mathbb{R}\times\sigma$ with σ a 3-dimensional manifold spacelike according to $g_{\mu\nu}$, eventually non-compact. The embedding of σ in M is described by the following:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_i N^i)\,dt^2 + 2N_i\,dx^i dt + h_{ij}\,dx^i dx^j \tag{44}$$

One could argue that, by our liberty of redefining the coordinates, it is possible to find a simpler coordinate-chart where the metric would take the simple diagonal form $\mathrm{diag}(1, h'_{ij})$ also called Gaussian coordinates, but we will see that keeping the non-dynamical modes $N(t)$ and $N_i(t, x_i)$ is more insightful.

$N(t)$ is called the time-lapse function and is the ratio of our artificial time t introduced by the foliation to the proper time of an observer whose geodesic follows a curve of constant x_i. N_i are called the time-shift functions and represent the deviation to the geodesics of the lines of constant x_i under an infinitesimal evolution $t \to t + dt$.

Expressing the previous action in terms of this new variables gives:

$$S = \int d^3x\,dt\,N\sqrt{h}\,(K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda) + S_{\text{matter}} \tag{45}$$

where ${}^3R$ is the remnant of the intrinsic curvature of M on Σ, and K_{ij} is the extrinsic curvature induced by the embedding and defined by:

$$K_{ij} = \frac{1}{2N}(-\frac{\partial h_{ij}}{\partial t} + 2D_{(i}N_{j)}) \tag{46}$$

with D the covariant derivative compatible with the metric h_{ij}, $(\dots)$ implies a symmetrization on the indexes and $K = K_{ij}h^{ij}$ is the trace of this curvature. Without losing generality, let's represent the matter content of our universe by a unique scalar field $\phi(t, x^i)$. The dynamical modes are ϕ with conjugate momentum:

$$\pi_\phi = \frac{\delta S}{\delta\dot{\phi}} = \frac{\sqrt{h}}{N}(\dot{\phi} - N^i\frac{\partial\phi}{\partial x^i}) \tag{47}$$

and the h_{ij} with momentum:

$$\pi_{ij} = \frac{\delta S}{\delta\dot{h}_{ij}} = -\sqrt{h}(K^{ij} - h^{ij}K) \tag{48}$$

Performing a Legendre transformation allows us to write (45) as:

$$S = \int d^3x dt\,[\pi^{ij}\dot{h_{ji}} + \pi_\phi\dot{\phi} - N\mathcal{H} - N_i H^i] \tag{49}$$

where:

$$H = G_{ijkl}\pi^{ij}\pi^{kl} - M_p^4\sqrt{h}[\,{}^3R - 2\Lambda + \frac{1}{2}(\frac{\pi_\phi^2}{2h} + h^{ij}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j} + V(\phi))] \tag{50}$$

$G_{ijkl} = \frac{1}{2}h^{-\frac{1}{2}}(h_{ik}h_{jl} + h_{jk}h_{il} - h_{ij}h_{kl})$ is named the De-Witt metric (on the space of $h_{ij}(x^i)$) and,

$$H_i = -2D_j\pi^{ij} + h^{ij}\pi_\phi\frac{\partial\phi}{\partial x^j} \tag{51}$$

Thus, the previous functions N and N^i impose the constraints $H_i = 0$, a direct consequence of the invariance under 3-dim space diffeomorphisms, and $H = 0$: a quite general result, any theory that presents invariance under local time-reparametrization has a vanishing Hamiltonian. Indeed, if one starts with the most general action:

$$S' = \int dt(p\dot{q} - H(q, \dot{q})) \tag{52}$$

and, assuming the freedom of time redefinition, performs the transformation $dt \to dt' = N(t)dt$, giving:

$$S' = \int dt' \frac{1}{N}(p\dot{q}N - H) = \int dt'(p\dot{q} - \frac{H}{N}) \tag{53}$$

Now, varying S' with respect to N and choosing the gauge $N = 1$ holds $H = 0$.

Quantum cosmology is the quantum theory of the dynamical systems describing closed cosmologies (from now on, our previous σ needs to be closed), based on the equations outlined above. It aims at describing the dynamics of the early universe, maybe the only laboratory where quantum gravity could be tested. The central object of this framework is the wave-function of the universe Ψ, carrying the quantum description of the dynamical fields appearing in general relativity: gravitation and matter.

$\Psi(h_{ij}, \phi, \Sigma)$ gives the amplitude that the universe contains a 3-surface Σ on which the 3-metric is $h_{ij}(x)$ and the matter field configuration $\phi(x)$. It needs to satisfy the famous Wheeler-DeWitt equation:

$$H\,\Psi(h_{ij}, \phi) = [-G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} - M_p^4 \sqrt{h}({}^3R - 2\Lambda) + \mathcal{H}_{\text{matter}}]\,\Psi(h_{ij}, \phi) = 0 \tag{54}$$

Recalling that in quantum mechanics, the Hamiltonian is the generator of time translations, (54) implies that Ψ is independent of the time. However, there is no time in general relativity, at least no absolute one, and, from there, it is coherent that any physical observable or any object that could lead to observables, like our wave-function Ψ, should not depend on our arbitrary choice of foliation. The interpretation of this equation remains an open question, but an easy picture is to see it as a consequence of the non existence of clocks exterior to the universe.

On the technical level, this equation is a functional differential equation of second-order non-polynomial and non analytic in the coefficients. So far, it has remained unsolved, no peculiar solutions has ever been found. It is even worst, no one has any idea about the type of Hilbert space capable of holding this H operator.

To circumvent these technical obstacles and make some use of the WdW equation, the most popular approximation is the mini-superspace approximation where one reduces the degrees of freedom from an infinite number, ie. defining $h_{ij}(t_0, x^i)$ and $\phi(t_0, x^i)$ everywhere on a space-like hypersurface, to a finite number; for instance a scale factor $a(t_0)$ and a scalar field $\phi(t_o)$ under the hypothesis of homogeneity. This field can be seen as an hypothetical dilaton, and from there study the likelihood of an inflating scenario emerging from different initial conditions.

So far, those are probably the most useful results extracted from simplified models in quantum cosmology: one looks at the peaks of the universe wave-function (localized at the classical solutions of Einstein equations) and then calculates the probability of this configuration to lead to a universe similar to the one we observe today, deciding of the relevance of this peculiar set of initial conditions (see [15] for detailed examples).

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